University of Liverpool

The Enumerative Geometry of Double Covers of Curves

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by

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Abstract

Let $\mathcal{A}dm(g,h)_{2m}$ be the space of admissible double covers $C \to D$ of curves of genus g and h, with all the ramification and branch points of C and D marked, and where the covering involution permutes an extra set of $2m$ marked points of C pairwise. For each $0 \le n \le 2g+2-4h$ there is a natural map $\phi_n \colon \mathcal{A}dm(g,h)_{2m} \to \overline{\mathcal{M}}_{g,n+2m}$ mapping the admissible cover $C \to D$ to the stabilization of the source curve C together with the $2m$ points and the first n ramification points.

In this thesis we will study classes of the form $[\phi_n(\mathcal{A}dm(g,h)_{2m})]$ in the Chow ring of $\overline{\mathcal{M}}_{g,n+2m}$. We will derive a formula for the intersection of any such class with the class of any decorated stratum class of $\overline{\mathcal{M}}_{g,n+2m}$ in Chapter 2. In Chapter 3 we will use this formula to compute the class $[\phi_n(\mathcal{A}dm(g,h)_{2m})]$ in terms of bases of decorated stratum classes for low values of g, h, n and m. In particular we give explicit expression in terms of decorated stratum classes of the class $[\phi_0(\mathcal{A}dm(4,1))]$ of the locus of bielliptic curves of genus 4 and the class $[\phi_0(\mathcal{A}dm(5,0))]$ of the locus of hyperelliptic curves of genus 5.

In Chapter 4 we will prove that for $g + 2m \ge 12$ and $g \ge 2$ the class $[\phi_n(\mathcal{A}dm(g,1)_{2m})]$ is not contained in the tautological ring. For $g + 2m = 12$ and $g \ge 2$ we will show that the same result holds on the moduli space $\mathcal{M}_{g,n+2m}$ of smooth curves.

Data

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Introduction

Classically enumerative geometry aims to count the number of objects of a certain kind, through geometric methods, whenever there are only a finite number of such objects. An archetypal example of the questions it tries to answer is

"How many conics in the plane pass through 5
given points in general position in the plane?"

To solve this question one possible strategy is to study the space X of conics in the plane. It turns out that the dimension of X is 5. For each point P in the plane there is a closed subspace Y_P of X of conics passing through P. The space Y_P defines a codimension one condition. Given 5 points $P_1, ..., P_5$ in general position in the plane the intersection between the spaces $Y_{P_1}, ..., Y_{P_5}$ defines the space of those conics C in X going through the points $P_1, ..., P_5$. Because the points are in general position each time we intersect with one of the spaces Y_{P_i} the dimension drops by 1. We deduce by this argument that the answer to the question posed above is a finite natural number. (in fact it turns out that the space X is linear, all the subspaces Y_{P_i} are also linear and therefore the answer is 1).

In enumerative geometry we then try to study moduli spaces X parameterizing some kind of objects we are interested in through the intersection theory of X. In other words we will try to understand the Chow ring $A^\bullet(X)$ or alternatively the cohomology ring $H^\bullet(X)$ of this moduli space. If we have some interesting locus $Y \subset X$ of objects satisfying a certain condition we want to have an understanding of the class $[Y] \in A^\bullet(X)$. To get a finite answer we can then intersect $[Y]$ with other interesting classes in $A^\bullet(X)$.

Suppose that Y_1 and Y_2 are two loci such that their intersection is a finite number of points. If Y_1' is a locus rationally equivalent to Y_1 we would like the intersection of Y_1' with Y_2 to be the same number of points as that of the intersection between Y_1 with Y_2. To ensure this we would like the space X to be complete (or compact). In cases where we are dealing with a noncompact moduli space X we often try to find some compactification of X. There are many possible ways to compactify a space X, however to be able to compute the number of intersection points within the compactification $\bar{X}$ of X it is crucial to have a some modular interpretation of the objects parametrized by $\bar{X}$.

A space that has been of particular interest in enumerative geometry is the moduli space of smooth curves $\mathcal{M}_g$ and its compactification $\overline{\mathcal{M}}_g$ by means of stable curves. The enumerative geometry of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ were first studied in the seminal paper [Mum83] and the field has seen a dramatic expansion since then. In this thesis we shall concern ourselves with $\overline{\mathcal{M}}_g$ and its intersection theory. When studying this intersection theory it turns out to be natural to study the moduli space $\overline{\mathcal{M}}_{g,n}$ parameterizing stable n-pointed curves as well.

Considered as a scheme the moduli space of curves is not smooth however when given the structure of a Deligne-Mumford stack it is smooth. An intersection theory for smooth complete Deligne-Mumford stacks was developed in [Vis89].

A thorough study of the Chow ring $A^\bullet(\overline{\mathcal{M}}_g)$ of the moduli space of stable curves was first done in [Fab90a] where the Chow ring $A^\bullet(\overline{\mathcal{M}}_3)$ of the moduli space of stable genus 3 curves is completely determined and in [Fab90b] where the groups $A^k(\overline{\mathcal{M}}_4)$ are determined for $k = 1$ and $k = 2$. Sadly it has turned out that $A^\bullet(\overline{\mathcal{M}}_{g,n})$ is rather hard to understand in general. However there is a well understood subring $R^\bullet(\overline{\mathcal{M}}_{g,n}) \subset A^\bullet(\overline{\mathcal{M}}_{g,n})$ called the tautological subring (see [Fab01]). This subring was first introduced in [Mum83] and a modern definition of this ring was given in [FP05]. In particular it has been shown in [GP03] that an *additive* set of generators for $R^\bullet(\overline{\mathcal{M}}_{g,n})$ is given by the so called decorated boundary strata and there is a combinatorial description of the intersection between two decorated boundary strata in terms of decorated boundary strata. Computer programs have since been developed (see [Pix] and [Yan08]) that can list a set of generators in terms of decorated boundary strata for the tautological ring and intersect decorated boundary strata.

Computing the set of relations between tautological classes has proven to be harder. A set of relations for the tautological ring $R^\bullet(\mathcal{M}_g)$ was conjectured by Faber and Zagier in 2000 and first proven to hold in [PP13]. These relations have been extended in [Pix13] to the tautological ring $R^\bullet(\overline{\mathcal{M}}_{g,n})$ of the moduli space of stable curves, they were proven to hold in cohomology in [PPZ16] and in Chow in [Jan15]. It has been conjectured that these relations form a complete set of relations between the generators of $R^\bullet(\overline{\mathcal{M}}_{g,n})$. The complete set of extended Faber-Zagier relations between a generating set of decorated boundary strata can also be computed by the computer program [Pix].

Since the moduli space $\overline{\mathcal{M}}_{g,n}$ is nonsingular as a stack the intersection pairing $H^k(\overline{\mathcal{M}}_{g,n}) \otimes H^{2\dim\overline{\mathcal{M}}_{g,n}-k}(\overline{\mathcal{M}}_{g,n}) \to \mathbb{C}$ is perfect. The Gorenstein conjecture asks whether the induced pairing remains perfect when restricted to the tautological subring. (see [Fab01]). As was first shown in [PT14] the Gorenstein conjecture is false in general, however for many low values of g, n and k it is known to hold. In cases where the Gorenstein conjecture holds the perfect pairing is often a powerful tool in intersection theoretic computations.

In this thesis we will study the enumerative geometry of loci $\mathcal{Y} \subset \overline{\mathcal{M}}_g$ of curves C admitting a 2-to-1 map to a curve D of a specified genus h. When the class $[\mathcal{Y}]$ of such a locus is contained in the tautological ring we will try to describe $[\mathcal{Y}]$ in terms of decorated boundary strata. In the case where $h = 0$ this locus is (a compactification of) the locus of hyperelliptic curves $\overline{\mathcal{H}}_g$. The hyperelliptic locus played an important role in the proof that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 23$ (see [HM82]). It turns out that all classes of hyperelliptic loci are tautological, this was proven in [FP05].

Over the years there has been an extensive effort to compute classes of such 2-to-1 covers. One of the simplest nontrivial examples of a computation of this kind which can be made is to compute the class $[\overline{\mathcal{H}}_3]$ of the locus of hyperelliptic curves in genus 3. For example it is shown in [HM98, Section 3.H] that

$$\begin{aligned}[\overline{\mathcal{H}}_3] &= 9\lambda - \delta_0 - 3\delta_1 \\ &= \frac{3}{4}\kappa_1 - \frac{1}{4}\delta_0 - \frac{7}{4}\delta_1 \in A^1(\overline{\mathcal{M}}_3)\end{aligned}$$

where the computation is done using test curves. The class $[\overline{\mathcal{H}}_4]$ of the hyperelliptic locus in genus 4 is computed in [FP05, Proposition 5]. More recently a recursive formula in terms of decorated boundary strata for the class $[\overline{\mathcal{H}}_{2,n,0}]$ of of the hyperelliptic locus in $A^\bullet(\overline{\mathcal{M}}_{2,n})$ with n Weierstrass points marked is studied in [CT17]. The class $[\overline{\mathcal{B}}_3]$ of the locus of bielliptic curves is studied in [FP15].

Apart from their intrinsic enumerative value, determining these classes can be valuable to study the effective and nef cones of the moduli space of curves. For classes in codimension 1 this has played an important role in studying the birational geometry of $\overline{\mathcal{M}}_{g,n}$. For example in [CC15, Theorem 5.6 and Remark 4.4] it is shown that the locus of hyperelliptic curves is an extremal codimension two cycle in $\overline{\mathcal{M}}_4$ but the locus $[\overline{\mathcal{B}}_3]$ of bielliptic curves of genus 3 is not extremal.

Outline

In Chapter 1 we will quickly introduce the basic definitions and statements from intersection theory and the theory of stacks needed later in this thesis. This chapter is intended to collect in one place the theory we shall use later. In particular we will define the moduli space of admissible double covers, give a statement of the excess intersection formula and introduce the tautological ring of the moduli space of curves.

The space of admissible double covers $\mathscr{A}dm(g,h)$ gives a compactification of the moduli space of double covers of smooth curves with the following modular interpretation: it parameterizes 2-to-1 maps of stable genus g curves to stable genus h curve where the ramification and the branch locus of the cover are ordered. From Riemann-Hurwitz it follows that the number of ramification points of such a cover is $2g+2-4h$. There are two natural maps going from $\mathscr{A}dm(g,h)$ that will be important throughout this thesis. The source map $\phi\colon \mathscr{A}dm(g,h) \to \overline{\mathcal{M}}_{g,2g+2-4h}$ takes the source curve C of an admissible cover $C \to D$ together with the marked ramification points. In the literature this map is often composed with the forgetful map $\overline{\mathcal{M}}_{g,2g+2-4h} \to \overline{\mathcal{M}}_{g,n}$ forgetting some (or all) of the ramification points and stabilizing the curve and we will denote the composition $\mathscr{A}dm(g,h) \to \overline{\mathcal{M}}_{g,2g+2-4h} \to \overline{\mathcal{M}}_{g,n}$ by ϕ_n. There is also a natural morphism $\rho\colon \mathscr{A}dm(g,h) \to \overline{\mathcal{M}}_{h,2g+2-4h}$ that sends an admissible cover $C \to D$ to the target curve D together with the data of the branch points.

$$\begin{tikzcd}
\mathscr{A}dm(g,h) \arrow[r, "\phi"] \arrow[rr, bend left, "\phi_n"] \arrow[d, "\rho"] & \overline{\mathcal{M}}_{g,2g+2-4h} \arrow[r] & \overline{\mathcal{M}}_{g,n} \\
\overline{\mathcal{M}}_{h,2g+2-4h} & &
\end{tikzcd}$$

In Chapter 2 we will work out the intersection in the Chow ring $A^\bullet(\overline{\mathcal{M}}_{g,n})$ between a space of admissible double covers and a decorated boundary stratum. To set up notation and as a warm up for the later arguments we first follow [GP03] to derive a combinatorial algorithm for the intersection between two decorated boundary strata in terms of decorated boundary strata. The original work of this thesis then starts in Section 2.2. In Theorem 2.2.21 we will give a combinatorial description of the pullback of a space of admissible double covers along a generalized gluing morphism $\overline{\mathcal{M}}_A \to \overline{\mathcal{M}}_{g,n}$ (where A is a stable graph) in terms of spaces of admissible double covers of curves in $\overline{\mathcal{M}}_A$. In Theorem 2.2.26 we will then give a combinatorial

description of the pullpush
$$\rho_*\phi_n^*([A_\theta]) \in A^\bullet(\overline{\mathcal{M}}_{h,2g+2-4h})$$
for any decorated boundary strata $[A_\theta] \in A^\bullet(\overline{\mathcal{M}}_{g,n})$ in terms of decorated boundary strata of $\overline{\mathcal{M}}_{h,2g+2-4h}$. Theorem 2.2.21 and 2.2.26 can perhaps be seen as the main technical results of this thesis. We deduce as a corollary that these pushpulls $\rho_*\phi^*$ map from the tautological ring $R^\bullet(\overline{\mathcal{M}}_{g,2g+2-4h})$ into the tautological ring $R^\bullet(\overline{\mathcal{M}}_{h,2g+2-4h})$, a result that was expected but for which we know of no existing reference. To clarify these combinatorial formulas we will finish this chapter with a number of examples.

In Chapter 3 we explain several ways in which the combinatorial descriptions given in Theorems 2.2.21 and 2.2.26 can be used to obtain explicit descriptions of classes of spaces of admissible double covers $[\phi_n(\mathcal{Adm}(g,h))] \in A^\bullet(\overline{\mathcal{M}}_g)$ in terms of a basis (or of a generating set) of decorated boundary strata. We will compute several such classes explicitly. In particular we will give expressions in terms of decorated boundary strata for the class $[\overline{\mathcal{H}}_5] = [\phi_0(\mathcal{Adm}(5,0))]$ of the locus of hyperelliptic curves in genus 5 and for the class $[\overline{\mathcal{B}}_4] = [\phi_0(\mathcal{Adm}(4,1))]$ of the locus of bielliptic curves of genus 4. These classes have never been computed explicitly before.

In the literature the spaces of hyperelliptic curves and bielliptic curves are usually taken without considering all the ramification points to be marked. It is not entirely standard to consider these spaces as the image under the forgetful morphism $\overline{\mathcal{M}}_{g,2g+2-4h} \to \overline{\mathcal{M}}_g$ of a space of the form $\mathcal{Adm}(g,h)$. In particular when we compute $\overline{\mathcal{H}}_5$ and $\overline{\mathcal{B}}_4$ we really need to use the moduli space $\overline{\mathcal{M}}_{g,n}$ with $n = 2g+2-4h$ and cannot make the calculation directly on $\overline{\mathcal{M}}_g$.

In [FP05] it has been shown that the class $[\overline{\mathcal{H}}_g] := [\phi_0(\mathcal{Adm}(g,0))]$ of hyperelliptic curves is always tautological. A natural question is in which other cases the class $[\phi_n(\mathcal{Adm}(g,h))]$ lies in the tautological ring. In Chapter 4 we will give a partial answer to this question by extending a result of Graber and Pandharipande [GP03] to show that for $g \geq 12$ the class
$$[\overline{\mathcal{B}}_g] := [\phi_0(\mathcal{Adm}(g,1))] \in A^\bullet(\overline{\mathcal{M}}_g)$$
of the locus of bielliptic curves does not lie in the tautological ring. This also gives a new lower bound for g for which nontautological algebraic classes are known to exist in $H^\bullet(\overline{\mathcal{M}}_g)$.

– I –

Preliminaries

In this chapter we will define Deligne-Mumford stacks and introduce bivariant intersection theory on such stacks. The main purpose of this chapter is to establish notation and some conventions for the rest of the thesis. This chapter can be skipped on a first reading by the reader already familiar with the material and referred back to at a later point.

1.1 Stacks

We will give a brief introduction to (Deligne-Mumford) stacks and in particular to the moduli stack of stable curves and the stack of admissible double covers. This section is mainly meant to set up notation and recall some necessary definitions and not as a comprehensive introduction. A full introduction to the theory of (algebraic) stacks can for example be found in [Ols16].

Throughout this thesis we will work over $\mathbb{C}$ although most of the statements in this thesis are still true over other algebraically closed fields of characteristic 0. Most of the statements in this thesis are false or do not make sense when considered for fields of characteristic 2.

Definition 1.1.1. A category $\mathcal{X}$ together with a functor $p_{\mathcal{X}} : \mathcal{X} \to \mathcal{S}$ to a category $\mathcal{S}$ is called a category fibered in groupoids over $\mathcal{S}$ if

1. for every morphism $f \colon Y \to X \in \operatorname{Mor} \mathcal{S}$ and every $x \in \operatorname{Ob} \mathcal{X}$ such that $p_{\mathcal{X}}(x) = X$ there exists a morphism $\phi \colon y \to x \in \operatorname{Mor} \mathcal{X}$ such that $p_{\mathcal{X}}(\phi) = f$,

2. for every commutative diagram

$$\begin{tikzcd} Y \arrow[dr, "f"] & \\ & X \\ Z \arrow[uu, "g"] \arrow[ur, "h"'] & \end{tikzcd}$$

in $\mathcal{S}$ let $\phi\colon y \to x$, $\eta\colon z \to x \in \operatorname{Mor}\mathcal{X}$ with $p_{\mathcal{X}}(\phi) = f$ and $p_{\mathcal{X}}(\eta) = h$ then there exists a morphism $\zeta\colon z \to y \in \operatorname{Mor}\mathcal{X}$ such that $\phi \circ \zeta = \eta$ and such that $p_{\mathcal{X}}(\zeta) = g$.

Definition 1.1.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be categories fibered in groupoids over $\mathfrak{S}$. A morphism of categories fibered in groupoids is a functor $F\colon \mathcal{X} \to \mathcal{Y}$ such that F commutes with the projection, i.e. $p_{\mathcal{X}} = p_{\mathcal{Y}} \circ F$.

Remark 1.1.3. There is a 2-category whose objects consist of categories fibered in groupoids, whose 1-morphisms are morphisms between categories fibered in groupoids and whose 2-morphisms are natural transformations.

Notation 1.1.4. Let $\mathcal{X}$ be a category fibered in groupoids over $\mathcal{S}$ and let $X \in \operatorname{Ob}\mathcal{S}$. We will denote by $\mathcal{X}(X)$ the category whose objects are objects $x \in \operatorname{Ob}\mathcal{X}$ such that $p_{\mathcal{X}}(x) = X$ and whose morphisms are $\phi \in \operatorname{Mor}\mathcal{X}$ such that $p_{\mathcal{X}}(\phi) = \mathrm{id}_X$. We will sometimes refer to X as a base object (or base scheme if $\mathcal{S} = \mathrm{Sch}$).

Remark 1.1.5. If $\mathcal{X}$ is a category fibered in groupoids over $\mathcal{S}$, then for any $X \in \operatorname{Ob}\mathcal{S}$ the category $\mathcal{X}(X)$ is a groupoid.

Remark 1.1.6. The Yoneda embedding of a scheme S defines a category fibered in groupoids. We will always abuse notation and denote by S the stack associated to a scheme S by the Yoneda embedding.

The 2-Yoneda lemma states that if $\mathcal{X}$ is any category fibered in groupoids there is a correspondence between maps from a scheme S to $\mathcal{X}$ and objects of $\mathcal{X}$ over S. More precisely there is an equivalence of categories (see [Vis04, Section 3.6.2])

$$\operatorname{Hom}_{\mathrm{Sch}}(S, \mathcal{X}) \simeq \mathcal{X}(S).$$

Definition 1.1.7. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be categories fibered in groupoids over $\mathcal{S}$ and let $F\colon \mathcal{X} \to \mathcal{Z}$ and $G\colon \mathcal{Y} \to \mathcal{Z}$ be morphisms. *The fiber product* $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is defined as follows. Objects of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ over a base $X \in \operatorname{Ob}\mathcal{S}$ are triples (x, y, α) where $x \in \operatorname{Ob}\mathcal{X}(X)$, $y \in \operatorname{Ob}\mathcal{Y}(X)$ and α is an isomorphism $F(x) \to G(y)$. A morphism $(x', y', \alpha') \to (x, y, \alpha)$ of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a pair $(\beta_1\colon x' \to x, \beta_2\colon y' \to y)$ such that $G(\beta_2) \circ \alpha' = \alpha \circ F(\beta_1)$.

There are natural projections P_1, P_2 from $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ to $\mathcal{X}$ and $\mathcal{Y}$ respectively. The projection P_1 is defined by mapping (x, y, α) to x and (β_1, β_2) to β_1. The projection P_2 is defined analogously. By definition the diagram

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \arrow[r, "P_2"] \arrow[d, "P_1"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{X} \arrow[r, "F"'] & \mathcal{Z} \end{tikzcd}$$

2-commutes. That is there is a natural isomorphism $\kappa : F \circ P_1 \Rightarrow G \circ P_2$.

Remark 1.1.8. The fiber product satisfies the following *strict universal property.* Given another category fibered in groupoids $\mathcal{W}$, morphisms $U\colon \mathcal{W} \to \mathcal{X}$, $V\colon \mathcal{W} \to \mathcal{Z}$ and a natural isomorphism $\kappa'\colon F \circ U \Rightarrow G \circ V$, then there exists a unique morphism $Q\colon \mathcal{W} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ such that $P_1 \circ Q = U$, $P_2 \circ Q = V$ and such that the natural isomorphism $\kappa'\colon F \circ U \Rightarrow G \circ V$ is the one determined by $\kappa\colon F \circ P_1 \Rightarrow G \circ P_2$. In other words we have the following diagram

$$\begin{tikzcd}
\mathcal{W} \arrow[drr, bend left, "V"] \arrow[dr, dashed, "\exists ! Q"] \arrow[ddr, bend right, "U"'] & & \\
& \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \arrow[r, "P_2"] \arrow[d, "P_1"] \arrow[dr, phantom, "\overset{\kappa}{\Rightarrow}"] & \mathcal{Y} \arrow[d, "G"] \\
& \mathcal{X} \arrow[r, "F"'] & \mathcal{Z}.
\end{tikzcd}$$

Stable Curves and Admissible Double Covers

All our categories fibered in groupoids will be fibered over $\mathrm{Sch}_{\mathbb{C}}$ for some algebraically closed field $\mathbb{C}$ of characteristic 0. We will now introduce the two examples of categories fibered in groupoids we will study in this thesis.

Definition 1.1.9. Let C/k be a scheme of finite type and pure dimension 1. A closed point $P \in C$ is called a *node* if the complete local ring $\hat{\mathcal{O}}_{C,P}$ is isomorphic to $k[[x,y]]/(xy)$. The scheme C is called a *nodal curve* if every point of C is either smooth or a node.

Definition 1.1.10. A *family of stable n-pointed curves* $\pi\colon C \to S$, $\{\sigma_i\}_{1\leq i \leq n}$ *of genus* g over a base scheme B is defined (see for example [BCE+, Example 2.10]) as a proper flat morphism $\pi\colon C \to B$ of finite presentation together with a collection of disjoint sections $\sigma_i\colon B \to C$, such that for each geometric point $s \in S$:

1. the fiber $\pi^{-1}(s)$ is a connected nodal curve of arithmetic genus g,
2. the points $\sigma_i(s)$ are nonsingular points of $\pi^{-1}(s)$,
3. each irreducible component C_j of $\pi^{-1}(s)$ with geometric genus $g(C_j)$ contains at least $3 - 2g(C_j)$ sections or points where it meets other components or itself.

Definition 1.1.11. *The moduli space of curves* $\overline{\mathcal{M}}_{g,n}$ is the category fibered in groupoids over Sch_k whose objects over a base B are families of stable n-pointed curves of arithmetic genus g over B and whose morphisms from $\pi'\colon C' \to B'$, $\{\sigma'_i\}_{1\leq i\leq n}$ to $\pi\colon C \to B$, $\{\sigma_i\}_{1\leq i\leq n}$ over $f \in \operatorname{Mor}\mathrm{Sch}_k$ are Cartesian diagrams

$$\begin{tikzcd}
C' \arrow[r, "g"] \arrow[d, "\pi'"'] & C \arrow[d, "\pi"] \\
B' \arrow[r, "f"'] & B
\end{tikzcd}$$

such that $\sigma_i \circ f = g \circ \sigma'_i$.

Definition 1.1.12. A *family of admissible* $2m$*-marked double covers* $f\colon S \to T$ over B is the data of two stable pointed curves $(T, \{\sigma_i\} \cup \{\tau_j\}_{1\leq j\leq m})$ and $(S, \{\tilde{\sigma_i}\} \cup \{\tilde{\tau}_k\}_{1\leq k\leq 2m})$ over B and a finite map $f\colon S \to T$ of generic degree 2 over B such that:

1. Every node of S maps to a node of T and the local picture of $S \to T \to B$ at a point of S (not necessarily geometric) over a node of T is isomorphic to

$$\operatorname{Spec} A[\xi, \eta]/(\xi\eta - a) \to \operatorname{Spec} A[x, y]/(xy - a^2) \to \operatorname{Spec} A$$

   where $f^*x = \xi^2$ and $f^*y = \eta^2$.

2. the restriction to the smooth locus $f^{sm} \colon S^{sm} \to T^{sm}$ is ramified exactly at the σ_i and we have $\sigma_i = f \circ \tilde{\sigma}_i$,

3. we have $f \circ \tilde{\tau}_{2j-1} = \tau_j = f \circ \tilde{\tau}_{2j}$,

We call S the source curve and T the target curve of the admissible cover. Note that an admissible cover induces an involution ι on S fixing the sections $\tilde{\sigma}_i$ and permuting the $\tilde{\tau}_k$ pairwise. If $g(T) = 0$ we say that the cover is an admissible hyperelliptic cover, if $g(T) = 1$ we say that the cover is an admissible bielliptic cover.

Remark 1.1.13. By repeated application of the Riemann-Hurwitz formula we see that the number of smooth ramification and branch points of an admissible cover $S \to T$ over $\mathbb{C}$ is $2g(S) + 2 - 4g(T)$.

Definition 1.1.14. *The space of admissible double covers* $\mathscr{Adm}(g, h)_{2m}$ (in this thesis sometimes simply called the space of admissible covers) is the category fibered in groupoids whose objects over a scheme B are admissible $2m$-marked double covers $(f \colon S \to T)$ with $p_a(S) = g$ and $p_a(T) = h$ and whose morphisms

$$h \colon (f' \colon S' \to T') \to (f \colon S \to T)$$

over $B' \to B$ are pairs of morphisms $h_1 \colon S' \to S$ and $h_2 \colon T' \to T$ of stable curves such that $f \circ h_1 = h_2 \circ f'$.

Remark 1.1.15. Admissible double covers of genus 0 curves were first defined in [Bea77] and then generalized to any degree in [HM82, page 57]). A modern general definition is given in [ACV03, Definition 4.3.1]. In this thesis we will only study admissible double covers. Our definition is slightly nonstandard in that we require *all* of the ramification points to be marked and in that we allow for some additional marked points $\tilde{\tau}_k$, τ_j of S and T outside of the branch/ramification locus of the cover.

There exists some dissatisfaction with the space of admissible covers as the moduli problem is not determined by the geometric objects it parameterizes (in particular note the first condition of Definition 1.1.12 is not just a condition on the geometric points). For a discussion on this, and possible resolutions, see [ACV03].

Definition 1.1.16. We define the *source map* $\phi \colon \mathscr{Adm}(g, h)_{2m} \to \overline{\mathcal{M}}_{g,2g+2-4h+2m}$ as the functor that sends an admissible cover $S \to T$ to the source curve $(S, \{\tilde{\sigma}_i\} \cup \{\tilde{\tau}_k\}_{1 \le k \le 2m})$.

Likewise we define the *target map* $\rho \colon \mathscr{Adm}(g, h)_{2m} \to \overline{\mathcal{M}}_{h,2g+2-4h+m}$ as the map sending an admissible cover $S \to T$ to the target curve $(T, \{\sigma_i\} \cup \{\tau_j\}_{1 \le j \le m})$.

Notation 1.1.17. We will denote by $\overline{\mathcal{M}}_{g,n}^D$ the moduli stack parameterizing étale double covers

$$f \colon (C_1; y_{1,1}, ..., y_{1,n}) \cup (C_2; y_{2,1}, ..., y_{2,n}) \to (C; y_1, ..., y_n)$$

mapping a disconnected *ordered* pair of isomorphic stable curves C_1 and C_2 2-to-1 to a stable curve $C \simeq C_1$ such that $f^{-1}(y_i) = (y_{i,1}, y_{i,2})$. A morphism of objects $(f\colon C_1 \cup C_2 \to C) \to (g\colon D_1 \cup D_2 \to D)$ in $\overline{\mathcal{M}}_{g,n}^D$ is a triple of morphisms $C_1 \to D_1, C_2 \to D_2, C \to D$ commuting with the covering maps.

The space $\overline{\mathcal{M}}_{g,n}^D$ is isomorphic to $\overline{\mathcal{M}}_{g,n}$, however we will later make use of the different modular interpretation provided by this definition.

Deligne-Mumford Stacks

We will now quickly define what it means for a category fibered in groupoids to be a stack.

Definition 1.1.18. Let C be a category. A *Grothendieck topology* on C consists of a set $\operatorname{Cov}(X)$ of coverings $\{\mathcal{X}_i \to X\}_{i \in I}$ for each $X \in \operatorname{Ob} C$ such that:

1. if $Y \to X$ is an isomorphism in C then $\{Y \to X\} \in \operatorname{Cov}(X)$,
2. if $\{X_i \to X\}_{i \in I} \in \operatorname{Cov}(X)$ and Y is any morphism in C, then the fiber products $X_i \times_X Y$ exist and
$$\{X_i \times_X Y \to Y\}_{i \in I} \in \operatorname{Cov}(Y),$$
3. if $\{X_i \to X\}_{i \in I} \in \operatorname{Cov}(X)$ and for each i we are given $\{Y_{ij} \to X_i\}_{j \in J_i} \in \operatorname{Cov}(X_i)$ then the composition
$$\{Y_{ij} \to X_i \to X\}_{i \in I, j \in J_i} \in \operatorname{Cov}(X).$$

A category together with a Grothendieck topology is called a *site*.

Example 1.1.19. The *(big) étale site of schemes over* $\mathbb{C}$ is the category $\mathrm{Sch}_{\mathbb{C}}$ of schemes over $\mathbb{C}$ together with the Grothendieck topology given by letting $\operatorname{Cov}(X)$ be the set of collections of morphisms $\{X_i \to X\}_{i \in I}$ (over $\mathbb{C}$) such that each morphism $X_i \to X$ is étale and the map
$$\coprod_{i \in I} X_i \to X$$
is surjective.

Definition 1.1.20. A category $\mathcal{X}$ fibered in groupoids over the big étale site Sch_C is a *stack over* $\mathbb{C}$ if for any étale covering $\{T_i \to T\}$ of schemes (over $\mathbb{C}$) we have

1. *Morphisms glue.* Let a, b be objects over T and let $\phi_i\colon a|_{T_i} \to b$ be morphisms over $T_i \to T$ such that $\phi_i|_{T_{ij}} = \phi_j|_{T_{ij}}$ where $T_{ij} := T_i \times_T T_j$. Then there exists a unique morphism $\phi\colon a \to b$ over the identity with $\phi|_{T_i} = \phi_i$. In other words we require that the solid commutative diagram

$$\begin{tikzcd}
 & a|_{T_i} \arrow[dr] \arrow[drr, "\phi_i"] & & \\
a|_{T_{ij}} \arrow[ur] \arrow[dr] & & a \arrow[r, dashed, "\phi"] & b \\
 & a|_{T_j} \arrow[ur] \arrow[urr, "\phi_j"'] & &
\end{tikzcd}$$

can be completed in a unique way by the dashed arrow $\phi\colon a \to b$.

2. *Objects glue.* For objects a_i over T_i with isomorphisms $\alpha_{ij}\colon a_i|_{T_{ij}} \to a_j|_{T_{ij}}$ over $\mathrm{id}_{T_{ij}}$ satisfying the cocycle condition $\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik}$ on T_{ijk} then there exists a unique object a over T and isomorphism $\phi_i\colon a|_{T_i} \to a_i$ over id_{T_i} such that $\alpha_{ij} \circ \phi_i|_{T_{ij}} = \phi_j|_{T_{ij}}$. In other words whenever we have a solid diagram

$$\begin{tikzcd} & & a_i \arrow[dr, dashed] & \\ a_i|_{T_{ij}} \arrow[r, "\alpha_{ij}"] & a_j|_{T_{ij}} \arrow[ur] \arrow[dr] & & a \\ & & a_j \arrow[ur, dashed] & \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} & T_i \arrow[dr] & \\ T_{ij} \arrow[ur] \arrow[dr] & & T \\ & T_j \arrow[ur] & \end{tikzcd},$$

where the α_{ij} satisfy the cocycle condition, the diagram can be completed with an object a over T.

Definition 1.1.21. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is called *representable* if for any scheme S and any map $S \to \mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y}} S$

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Y}} S \arrow[r] \arrow[d] & S \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

is a scheme.

Definition 1.1.22. Let $\mathbf{P}$ be a property of maps of schemes stable under pullback. We say that a representable morphism of stacks $\mathcal{X} \to \mathcal{Y}$ has property $\mathbf{P}$ if for any scheme S and any morphism $S \to \mathcal{Y}$ the map $\mathcal{X} \times_{\mathcal{Y}} S \to S$ has property $\mathbf{P}$.

Definition 1.1.23. A *Deligne-Mumford stack* is a stack $\mathcal{X}$ with the following properties:

1. any morphism from a scheme S into $\mathcal{X}$ is representable,
2. there exists an étale covering $S \to \mathcal{X}$ of $\mathcal{X}$ by a scheme S. The scheme S is called an *atlas*.

Definition 1.1.24. A Deligne-Mumford stack $\mathcal{X}$ is called *Noetherian* if there exists an atlas $S \to \mathcal{X}$ with S Noetherian. A morphism of Deligne-Mumford stacks $\mathcal{X} \to \mathcal{Y}$ is called of finite type if there exists an atlas $S \to \mathcal{X}$ such that $S \to \mathcal{Y}$ is of finite type. We will assume all our stacks to be Noetherian and of finite type.

Definition 1.1.25. A morphism of Noetherian Deligne-Mumford stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is *separated* if for any complete discrete valuation ring R and any commutative diagram

$$\begin{tikzcd} & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] \arrow[r] & \mathcal{Y} \end{tikzcd}$$

any isomorphism between the restrictions of g_1 and g_2 to the generic point of $\mathrm{Spec}(R)$ can be extended to an isomorphism between g_1 and g_2.

Definition 1.1.26. A morphism of Noetherian Deligne-Mumford stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is *proper* if it is separated, of finite type and for any complete valuation ring R with field of fractions K and any solid commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r, dotted] \arrow[d, dotted] & \mathrm{Spec}(K) \arrow[r, "g"] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathrm{Spec}(R') \arrow[r, dotted] \arrow[urr, dotted] & \mathrm{Spec}(R) \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

there exists a finite extension K' of K such that the morphism $\mathrm{Spec}(K') \to \mathcal{X}$ induced by g extends to $\mathrm{Spec}(R')$, where R' is the integral closure of R in K'.

Remark 1.1.27. These are the valuative criteria for separatedness and properness. For representable morphisms of stacks these definitions coincide with that given by Definition 1.1.22 and the definitions of separated and proper schemes. Other definitions can be given but in the setting of moduli spaces Definitions 1.1.25 and 1.1.26 turn out to be the most useful characterization (see [DM69]).

Definition 1.1.28. A Deligne-Mumford stack over $\mathbb{C}$ is *complete* if it is proper over $\mathbb{C}$.

Theorem 1.1.29. The categories fibered in groupoids $\overline{\mathcal{M}}_{g,n}$ and $\mathscr{Adm}(g,h)_{2m}$ are smooth complete Deligne-Mumford stacks.

Proof. The first of these is well known, a proof can for example be found in [ACG11]. For $\mathscr{Adm}(g,h)_{2m}$ see [Moc95, Paragraph 3.22] (also see [ACV03, Section 4]). ☐

Remark 1.1.30. The source map $\phi\colon \mathscr{Adm}(g,h)_{2m} \to \overline{\mathcal{M}}_{g,2g+2-4h+2m}$ from Definition 1.1.16 is representable and a closed embedding.

Definition 1.1.31. Let $(\mathcal{X}, p)$ be a category fibered in groupoids (over Sch_k). A *quasi-coherent sheaf* $\mathcal{F}$ on $\mathcal{X}$ consists of the datum (see for example [ACG11, Definition 13.2.1]):

1. a quasi-coherent sheaf $\mathcal{F}_x$ on $p(x)$ for any $x \in \mathrm{Ob}\,\mathcal{X}$,
2. an isomorphism
$$\rho_\phi\colon f^*(\mathcal{F}_y) \to \mathcal{F}_x$$
for any morphism $\phi\colon x \to y \in \mathrm{Mor}\,\mathcal{X}$ over $f \in \mathrm{Mor}\,\mathrm{Sch}_k$,

such that for any pair of morphisms $\phi\colon x \to y$, $\eta\colon y \to z$ in $\mathrm{Mor}\,\mathcal{X}$ over f and g in $\mathrm{Mor}\,\mathrm{Sch}_k$ the diagram

$$\begin{tikzcd}
f^*g^*\mathcal{F}_z \arrow[r, "\sim"] \arrow[d, "f^*(\rho_\eta)"] & (gf)^*\mathcal{F}_z \arrow[d, "\rho_{\eta\circ\phi}"] \\
f^*\mathcal{F}_y \arrow[r, "\rho_\phi"] & \mathcal{F}_x
\end{tikzcd}$$

of isomorphisms of sheaves over $p(x)$ commutes.

Definition 1.1.32. Let $\mathcal{X}$ be a reduced Deligne-Mumford stack. We define a vector bundle on $\mathcal{X}$ as a locally free coherent sheaf on $\mathcal{X}$.

When $\mathcal{X}$ is disconnected we allow a vector bundle E on $\mathcal{X}$ to have a different rank on each connected component.

Example 1.1.33. Let $\mathcal{X}$ be a smooth Deligne-Mumford stack, the tangent bundle $\mathcal{T}_{\mathcal{X}}$ of $\mathcal{X}$ can be defined as follows (see for example [ACG11, Section 13.3]). Given an object $x \in \mathcal{X}(S)$ with $x\colon S \to \mathcal{X}$ an étale map. If $\xi\colon x \to y$ is a morphism of $\mathcal{X}$ lying over $f\colon S \to T$ the isomorphism $\mathcal{T}_x \to f^*\mathcal{T}_y$ is defined by the differential of f. This defines a vector bundle on $\mathcal{X}$.

Definition 1.1.34. Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibered in groupoids and let $\mathcal{F}$ be a quasi coherent sheaf on $\mathcal{Y}$. The *pullback* $F^*(\mathcal{F})$ is defined by:

1. for any $x \in \operatorname{Ob}\mathcal{X}$
$$(F^*\mathcal{F})_x = \mathcal{F}_{F(x)},$$
2. for any morphism $\xi\colon x \to y \in \operatorname{Mor}\mathcal{X}$ over $f \in \operatorname{Mor}\mathrm{Sch}_k$
$$\begin{aligned} &F^*(\rho_\xi)\colon f^*(F^*\mathcal{F})_y \to (F^*\mathcal{F})_x \\ &= \rho_{F(\xi)}\colon f^*(\mathcal{F}_{F(y)}) \to \mathcal{F}_{F(x)}. \end{aligned}$$

Example 1.1.35. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable regular embedding of smooth Deligne-Mumford stacks. As for schemes the normal bundle can be defined as $\mathcal{N}_f := \mathcal{T}_{\mathcal{Y}}/f^*\mathcal{T}_{\mathcal{X}}$.

1.2 Intersection Theory

In this thesis we shall use intersection theory in the sense of Fulton-MacPherson (see [Ful84]) generalized to Deligne-Mumford stacks (as in [Vis89]). We will now collect some results from bivariant intersection theory we shall use in this thesis.

Definition 1.2.1. Let $\mathcal{X}$ be a Deligne-Mumford stack. A *cycle of dimension* k on $\mathcal{X}$ is an element of the free abelian group $Z_k(\mathcal{X})$ generated by all integral closed substacks of dimension k.

The Chow group $A_k(\mathcal{X})$ is defined as the group of cycles of dimension k modulo rational equivalence. For more detail see [Vis89, Definition 3.4].

The Chow group with rational coefficients is defined as $A_k(\mathcal{X})_{\mathbb{Q}} := A_k(\mathcal{X}) \otimes \mathbb{Q}$. In this thesis all Chow groups will be taken with rational coefficients and we will simply write $A_k(\mathcal{X})$ for the Chow groups with rational coefficients.

Definition 1.2.2. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a separated dominant morphism of finite type of integral Deligne-Mumford stacks. Let $X \to \mathcal{X}$ and $Y \to \mathcal{Y}$ be étale covers with X and Y integral schemes. If f is representable we set
$$\deg(\mathcal{X}/\mathcal{Y}) := \deg(\mathcal{X} \times_{\mathcal{Y}} Y/Y).$$

If f is not representable we set
$$\deg(\mathcal{X}/\mathcal{Y}) := \deg(X/\mathcal{Y})/\deg(X/\mathcal{X}).$$

Proposition 1.2.3. We have

$$\deg(\mathcal{A}dm(g,h)_{2m}/\overline{\mathcal{M}}_{h,2g+2-4h+m}) = \begin{cases} (2^{2h}-1)2^{m-1} & \text{when } 2g+2-4h=0 \\ 2^{2h+m-1} & \text{when } 2g+2-4h>0 \end{cases} \tag{1.1}$$

and

$$\deg(\overline{\mathcal{M}}^D_{g,m}/\overline{\mathcal{M}}_{g,m}) = 1. \tag{1.2}$$

Proof. We restrict to the smooth locus. For a smooth curve T (over $\mathbb{C}$) of genus h, double covers of T with branch divisor b coincide with square roots of $\mathcal{O}_T(b)$. Given one such square root $\mathcal{R}$ all other square roots are given by sheaves of the form $\mathcal{R}\otimes\mathcal{L}$ with $\mathcal{L}^2=\mathcal{O}_T$. The Jacobian of T has $2^{2h}-1$ torsion points of order 2 and 1 torsion point of order 1. In the ramified case all these torsion points coincide with admissible double covers. However, we require the source curve of an admissible cover to be connected and in the unramified case one of the torsion points coincides with the unconnected double cover $T \amalg T \to T$.

If $m=0$ then the covering involution of the admissible cover $S\to T$ induces an automorphism of order 2 which disappears under the forgetful map $\mathcal{A}dm(g,h)\to\overline{\mathcal{M}}_{h,2g+2-4h}$. If $m\geq 1$ then the covering involution does not induce an automorphism on the admissible covers. However, for a smooth double cover $f\colon S\to T$ with m marked points $(p_1,\ldots,p_m)$ on T, there are, 2^{m-1} nonisomorphic ways of choosing $2m$ points $(\tilde{p},\ldots,\tilde{p}_{2m})$ in S such that $f(\tilde{p}_{2i})=f(\tilde{p}_{2i+1})=p_i$, we thus obtain Equation 1.1.

Equality 1.2 follows from the fact that $\overline{\mathcal{M}}^D_{g,n}\simeq\overline{\mathcal{M}}_{g,n}$ (see Notation 1.1.17). □

Definition 1.2.4. Let $f\colon\mathcal{X}\to\mathcal{Y}$ be a representable morphism of Deligne-Mumford stacks. A *bivariant class* α in $A^l(\mathcal{X}\to\mathcal{Y})$ associates to each pair (g,D), where $g\colon T\to\mathcal{Y}$ is a morphism with T a scheme and $D\in A_k(T)$, a class denoted $\alpha_g\cap D\in A_{k-l}(\mathcal{X}\times_{\mathcal{Y}}T)$ (or simply $\alpha\cap D$) such that the following conditions are satisfied:

1. Let $p\colon T_2\to T_1$ be a morphism of schemes and form the diagram

$$\begin{tikzcd} S_2 \arrow[r] \arrow[d, "q"] & T_2 \arrow[d, "p"] \\ S_1 \arrow[r] \arrow[d] & T_1 \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}.$$

   where both squares are fibered squares. If p is proper and $D\in A_k(T_2)$, then

$$\alpha_g\cap p_*(D) = q_*\alpha\cap D\in A_{k-l}(S_1).$$

   If p is flat of relative dimension n and $D\in A_k(T_1)$ then

$$\alpha_{q\circ p}\cap p^*D = q^*(\alpha_g\cap D)\in A_{k+n-l}(S_2).$$

2. Let $p\colon\mathcal{Z}_2\to\mathcal{Z}_1$ be a regular local embedding of Deligne-Mumford stacks of codimension e. Let $f\colon T_1\to\mathcal{Y}$ and $u\colon T_1\to\mathcal{Z}_1$ be morphisms with T_1 a scheme and form the fiber diagram

$$\begin{tikzcd} S_2 \arrow[r] \arrow[d] & T_2 \arrow[r] \arrow[d] & \mathcal{Z}_2 \arrow[d, "h"] \\ S_1 \arrow[r] \arrow[d] & T_1 \arrow[r, "u"] \arrow[d, "g"] & \mathcal{Z}_1 \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} & \end{tikzcd} .$$

Then, for all $D \in A_\bullet(T_1)$, we have

$$\alpha_{g\circ p} \cap h^! D = h^!(\alpha_g \cap D) \in A_{k-l-e}(S_2)$$

where $h^!$ is the Gysin homomorphism (see [Vis89, Definition 3.10])

Definition 1.2.5. We have 3 fundamental operations among bivariant classes:

1. *Product.* Let $f\colon \mathcal{X} \to \mathcal{Y}$, $g\colon \mathcal{Y} \to \mathcal{Z}$ be representable morphisms of Deligne-Mumford stacks and $\alpha \in A^{l_1}(f)$, $\beta \in A^{l_2}(g)$ then we define the product

$$\alpha \cdot \beta \in A^{l_1+l_2}(f \circ g)$$

by

$$(\alpha \cdot \beta) \cap D = \alpha \cap (\beta \cap D)$$

for any morphism $S \to \mathcal{Z}$ with S a scheme and any class $D \in A_\bullet(S)$.

2. *Proper pushforward.* Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ be representable morphisms of Deligne-Mumford stacks with f a proper. We define

$$f_*\colon A^\bullet(\mathcal{X} \to \mathcal{Z}) \to A^\bullet(\mathcal{Y} \to Z)$$

as follows. If $\alpha \in A^\bullet(fg\colon \mathcal{X} \to \mathcal{Z})$, $S \to \mathcal{Z}$ is a morphism with S a scheme, $h\colon \mathcal{X} \times_\mathcal{Z} S \to \mathcal{Y} \times_\mathcal{Z} S$ the induced morphism and $D \in A_\bullet(S)$, then

$$(f_*\alpha) \cap D := h_*(\alpha \cap D).$$

3. *Pullback.* If $f\colon \mathcal{X} \to \mathcal{Z}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are morphisms of Deligne-Mumford stacks with f representable, then we define

$$g^*\colon A^\bullet(\mathcal{X} \to \mathcal{Z}) \to A^\bullet(\mathcal{X} \times_\mathcal{Z} \mathcal{Y} \to \mathcal{Y})$$

by

$$(g^*\alpha) \cap D = \alpha \cap D$$

for any scheme S any morphism $S \to \mathcal{Y}$ and any class $D \in A_\bullet(S)$.

Notation 1.2.6. For any Deligne-Mumford stack $\mathcal{X}$ we set

$$A^\bullet(\mathcal{X}) := A^\bullet(\mathrm{id}\colon \mathcal{X} \to \mathcal{X}) \otimes \mathbb{Q}.$$

This is a graded ring under products and we will call it the *Chow ring (with rational coefficients)* of $\mathcal{X}$.

Proposition 1.2.7 (Poincaré duality)**.** If X is a smooth integral Deligne-Mumford stack then there is an isomorphism of groups

$$\begin{aligned} A^\bullet(\mathcal{X}) &\to A_\bullet(\mathcal{X}) \\ \alpha &\mapsto \alpha \cap [\mathcal{X}]. \end{aligned}$$

Proposition 1.2.8. If $\mathcal{X}$ is a Deligne-Mumford stack, V a vector bundle on $\mathcal{X}$ and c_i the i'th chern class, then $c_i(V) \in A^i(\mathcal{X})$.

Definition 1.2.9. A representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne-Mumford stacks is called *locally complete intersection morphism of relative dimension d, or simply lci morphism,* if it factors as a closed regular embedding i of some codimension e followed by a smooth morphism p of relative dimension $d+e$.

Definition 1.2.10. If $f\colon \mathcal{X} \to \mathcal{Y}$ is a representable lci morphism of Deligne-Mumford stacks with factorization $f = p \circ i$ then we define an element $[f] \in A^\bullet(f\colon \mathcal{X} \to \mathcal{Y})$ by $f \cap D = p^* i^!(D)$ (see [Ful84, Section 17.4]) where $i^!$ is the Gysin homomorphism as defined in [Vis89, Definition 3.10].

Proposition 1.2.11. The following facts are well known:

1. morphisms between smooth stacks are lci,
2. morphisms between complete stacks are proper.

Remark 1.2.12. One can define a normal bundle for lci morphisms of smooth Deligne-Mumford stacks in the same way as is done for schemes (see [Ful84, B.7.6]).

We now recall the excess intersection formula.

Proposition 1.2.13 (Excess intersection formula)**.** Consider the fibered square

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f'"] \arrow[d, "g'"] & \mathcal{Y} \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Z} \end{tikzcd} \tag{1.3}$$

and suppose all morphisms are representable and suppose f and f' are lci of codimension d and d' respectively. Let

$$E := g'^* N_f / N_{f'}$$

be the *excess bundle.* Then

$$g^*[f] = c_{d-d'}(E) \cdot [f'] \in A^d(\mathcal{X}' \to \mathcal{Y}').$$

Proof. See [Ful84, Theorem 17.4.1]. □

Corollary 1.2.14. Consider again the fibered square 1.3 and suppose g is proper. Let $D \in A_k(\mathcal{Y})$, with the notation of Proposition 1.2.13 we have

$$f^* g_*(D) = g'_*(c_{d-d'}(E) \cap f'^*(D)) \in A_{k-d}(\mathcal{X})$$

Proof. See [Ful84, Example 17.4.1.a]. □

1.3 The Tautological Ring of $\overline{\mathcal{M}}_{g,n}$

We want to study the enumerative geometry of the moduli space of curves. Classically this means that given a list of properties $\{P_i\}$ of curves of genus g such that there is only a finite amount of curves satisfying all of these conditions we would like to count, with the appropriate multiplicity, the amount of curves satisfying these conditions. We can make this precise as follows, each of the properties P_i cuts out a closed substack $\mathcal{P}_i \subset \overline{\mathcal{M}}_{g,n}$ and therefore defines an element of $A^\bullet(\overline{\mathcal{M}}_{g,n})$. The number

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod \mathcal{P}_i = p_* \left(\prod \mathcal{P}_i \right),$$

where p is the morphism $\overline{\mathcal{M}}_{g,n} \to \operatorname{Spec} \mathbb{C}$, is then the number of curves satisfying the properties P_i. Note that since we are working with the Chow ring with *rational coefficients* (which we do because we are dealing with stacks) this number might be a rational number, not just an integer.

We would then like to understand the Chow ring $A^\bullet(\overline{\mathcal{M}}_{g,n})$. It turns out that understanding this ring is very hard in general. Indeed it is not even finitely generated! (For example $A_0(\overline{\mathcal{M}}_{1,11})$ is infinite dimensional see [GP03]). Passing through the cycle map to the cohomology ring $H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ the dimension is at least finite, but understanding this ring is still considered to be very hard.

The idea is then to study a subring $R^\bullet(\overline{\mathcal{M}}_{g,n})$ called the tautological ring of the Chow ring $A^\bullet(\overline{\mathcal{M}}_{g,n})$ which we can understand and which we hope contains most classes which we are geometrically interested in. In Chapter 4 we will show that the locus of (admissible) bielliptic curves of genus $g \geq 12$ does not lie in this tautological ring indicating some of the limitations of this tautological ring.

Definition 1.3.1. The *gluing morphisms*

$$\xi \colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

is defined by gluing pairs of marked curves together in the last marked point of each curve. Similarly the *gluing morphism*

$$\xi^{\mathrm{irr}} \colon \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$$

is defined by gluing each curve to itself in the last two markings.

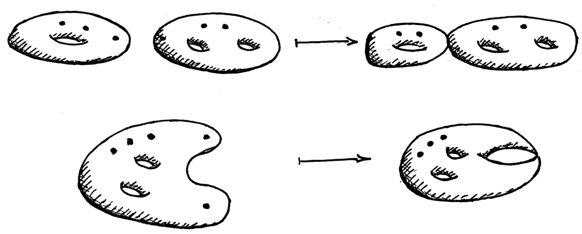


Figure 1.1: The gluing morphisms

The *forgetful morphism*

$$\pi \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

is defined by forgetting one of the markings (usually taken to be the last one) of each curve and stabilizing the curve by contracting any genus 0 component with only 2 marked points on it.

Definition 1.3.2. The tautological rings $R^\bullet(\overline{\mathcal{M}}_{g,n})$ can be defined (see [FP05]) as the minimal system of $\mathbb{Q}$-subalgebras of $\{A^\bullet(\overline{\mathcal{M}}_{g,n})\}$ closed under pushforward along the gluing morphisms and forgetful morphisms of Definition 1.3.1.

Notation 1.3.3. We will write ϕ_n for the composition $\mathcal{A}dm(g,h)_{2m} \to \overline{\mathcal{M}}_{g,2g+2-4h+2m} \to \overline{\mathcal{M}}_{g,n+2m}$ where the first map is the source map ϕ and the second map is a composition of forgetful morphisms.

We will write $\overline{\mathcal{H}}_{g,n,2m}$ for $\phi_n(\mathcal{A}dm(g,0)_{2m})$ and $\overline{\mathcal{B}}_{g,n,2m}$ for $\phi_n(\mathcal{A}dm(g,1)_{2m})$. We will drop m from the notation $\mathcal{A}dm(g,h)_{2m}$ when $m=0$.

Definition 1.3.4. Let $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism forgetting the last point and stabilizing and let $\sigma_i\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ be the section corresponding to the i'th marked point. Let ω_π be the dualizing sheaf of the map π. We define the *cotangent line bundle* to be

$$\mathbb{L}_i := \sigma_i^*(\omega_\pi)$$

and set

$$\psi_i := c_1(\mathbb{L}_i).$$

Let D_i be the divisor of $\overline{\mathcal{M}}_{g,n+1}$ corresponding to the image of σ_i and set $D = \sum_i D_i$. Let $\omega_\pi(D)$ be the log canonical sheaf. We also define the *Mumford-Morita-Miller κ classes*

$$\kappa_j := \pi_*(c_1(\omega_\pi(D))^{j+1}) = \pi_*(\psi_{n+1}^{j+1}).$$

Proposition 1.3.5. The ψ and κ classes lie in the tautological ring.

Remark 1.3.6. To understand the tautological ring $R^\bullet(\overline{\mathcal{M}}_{g,n})$ we need to know the following three things:

i a set of additive generators of $R^k(\overline{\mathcal{M}}_{g,n})$,

ii the space of all relations between these generators,

iii an algorithm for computing the product between any two such generators D_1 and D_2 of codimension k_1 and k_2 in terms of the generators of $R^{k_1+k_2}(\overline{\mathcal{M}}_{g,n})$.

Points 1 and 3 have been completely understood and will be treated in Section 2.1 of this thesis.

The space of relations between known sets of generators has been well studied and there are conjectures about possibly complete sets of generators. In particular Pixton has extended a set of relations by Faber and Zagier on the moduli space of smooth curves $\mathcal{M}_g$ to all of $\overline{\mathcal{M}}_{g,n}$ and conjectured that these generate all relations. Pixton has also written a program which computes all of these relations effectively.

We introduce some of the relations particularly useful later.

Example 1.3.7. Let $(12|34)$ be the divisor in $\overline{\mathcal{M}}_{0,4}$ of curves of genus 0 consisting of two irreducible components with the marked points 1 and 2 on one irreducible component and 3 and 4 on the other one. Define $(13|24)$ and $(14|23)$ similarly, then

$$(12|34) = (13|24) = (14|23) \in R^1(\overline{\mathcal{M}}_{0,4}).$$

This relation is called the *fundamental relation.*

Example 1.3.8. If $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism, then we have (see for example [ACG11, Equations 17.4.18 and 17.4.19])

$$\pi_*(\kappa_j) = \kappa_{j-1}$$
$$\pi^*(\kappa_j) = \kappa_j - \psi_{n+1}^j.$$

Proposition 1.3.9. On $\overline{\mathcal{M}}_{0,n}$ we have

$$\psi_1 = D_{1|23} \in R^1(\overline{\mathcal{M}}_{0,n})$$

where $D_{1|23}$ is the divisor of all genus 0 curves with the marking 1 on one irreducible component and the markings 2 and 3 on another.

The following is also a well known fact which will be useful in some of the computations in Chapter 3 and will play a pivotal role in Chapter 4.

Remark 1.3.10. For the intersection theoretic parts in Chapter 2 of this thesis we will always use the Chow ring $A^\bullet(\overline{\mathcal{M}}_{g,n})$. However in Chapter 3 we will sometimes need to pass through the cycle map $\mathrm{cyc}\colon A^\bullet(\overline{\mathcal{M}}_{g,n}) \to H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ and make computations in the cohomology ring instead. We define a *tautological cohomology ring* by setting

$$RH^{2\bullet} := \mathrm{cyc}(R^\bullet(\overline{\mathcal{M}}_{g,n})).$$

The main reasons for this are that we have a better understanding of the generators and relations of $H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ and $RH^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ than of $A^\bullet(\overline{\mathcal{M}}_{g,n})$ and $R^\bullet(\overline{\mathcal{M}}_{g,n})$ and that we can use Poincaré duality in $H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$. In Chapter 4 we will also exploit the odd cohomology of $\overline{\mathcal{M}}_{g,n}$.

We will not define cohomology for Deligne-Mumford stacks here because we will never need to work directly with the definition. For an overview see [AGV08, Section 2.2].

Proposition 1.3.11. Let $\mathcal{X}$ be a nonsingular complete Deligne-Mumford stack. Let $\Delta \subset \mathcal{X} \times \mathcal{X}$ be the diagonal. Given a homogeneous basis $\{e_i\}_{i\in I}$ for $H^\bullet(\mathcal{X})$ with dual basis $\{\hat{e}_i\}_{i\in I}$, the cohomology class of the diagonal can be written as

$$[\Delta] = \sum_{i\in I} (-1)^{\deg e_i} e_i \otimes \hat{e}_i.$$

– II –

Intersections on $\overline{\mathcal{M}}_{g,n}$

In this chapter we will give an algorithm for computing the intersection between any decorated stratum classes and any space of admissible double covers. We will also compute the pushforward of this intersection along the target map $\pi\colon \mathscr{A}dm(g,0)_{2m} \to \overline{\mathcal{M}}_{0,2g+2+m}$. To set up notation and because the argument with admissible covers is similar, we will start by computing the intersection between two decorated stratum classes. For this we follow [GP03] and [Yan08].

2.1 Boundary Strata

Definition 2.1.1. Recall that an *undirected finite graph* (or simply a *graph*) is a triple

$$(V, E, s\colon E \to \operatorname{Sym}^2 V)$$

where V is a finite set of vertices, E is a finite set of edges and s maps each edge into the second symmetric power of V thus assigning two vertices to every edge.

Also recall that a *path* between two vertices $v, v' \in V$ is a sequence of vertices

$$v = v_0, v_1, ..., v_n = v'$$

and a sequence of edges $e_1, ..., e_n$ such that $s(e_i) = v_{i-1}v_i$ for all i. A graph is said to be *connected* if there is a path between any pair of vertices $v, v' \in V$.

Definition 2.1.2. We define a *stable graph* Γ to be the data

$$\Gamma := (V, H, L, g\colon V \to \mathbb{Z}_{\geq 0}, \iota\colon H \to H, a\colon H \to V, \zeta\colon L \to V)$$

that satisfies the following conditions:

i ι is a fixed point free involution,

ii let E be the set of orbits of ι and let $s\colon E \to \operatorname{Sym}^2 V$ be the function induced by a on E; then (V, E, s) is a connected graph,

iii for each vertex $v \in V$ the stability condition $2g(v) - 2 + |a^{-1}(v)| + |\zeta^{-1}(v)| > 0$ is satisfied.

Notation 2.1.3. We call the elements of H *half edges* and the elements of L *legs*. We call g the *genus function* and say that $g(v)$ is the genus of the vertex v. For a vertex $v \in V$ we set $n(v) = |a^{-1}(v)| + |\zeta^{-1}(v)|$ the number of *half edges plus legs incident to* v. The *genus of a stable graph* Γ is defined to be

$$g(\Gamma) := \sum_{v \in V} g(v) + h^1(V, E, s).$$

where (V, E, s) is the connected graph induced by Γ. We will say that a stable graph Γ is n-pointed if $n = |L|$.

Example 2.1.4. Let C be a stable curve over $\operatorname{Spec} \mathbb{C}$. We associate a stable graph to C as follows. Let V be the set of irreducible components of C, let H be the set of sections of the normalization of C corresponding to the nodes of C, L the set of marked points of C, g the function which associates the geometric genus to each irreducible component of C, ι the involution identifying sections $h \in H$ mapping to the same node of C, a the map identifying sections with the irreducible component they lie on and ζ the map sending a marked point to its corresponding irreducible component. The graph $(V, H, L, g, \iota, a, \zeta)$ is stable and is called the *dual graph* of C.

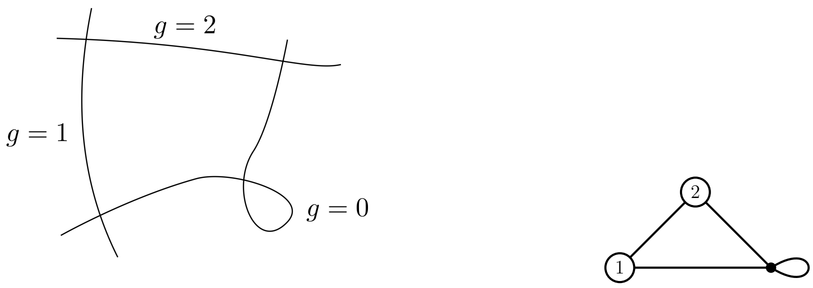


Figure 2.1: A curve and its dual graph. When the genus of a vertex is 0 we shall depict it as a black dot.

Definition 2.1.5. Let A and Γ be genus g stable graphs with set of legs $L_\Gamma = L = L_A$. An A-structure on Γ is a triple

$$(\alpha\colon V_\Gamma \twoheadrightarrow V_A,\ \beta\colon H_A \hookrightarrow H_\Gamma,\ \gamma\colon H_\Gamma \setminus \operatorname{Im} \beta \to V_A)$$

which satisfies the following conditions:

i the map β commutes with the involutions, i.e. $\beta \circ \iota_A = \iota_\Gamma \circ \beta$,

ii the map α respects the leg assignments, i.e. $\alpha \circ \zeta_\Gamma = \zeta_A$,

iii if $h \in \operatorname{Im} \beta$ and $a_\Gamma(h) = v$ then $a_A(\beta^{-1}(h)) = \alpha(v)$,

iv if $h \in H_\Gamma \setminus \operatorname{Im} \beta$ and $a_\Gamma(h) = v$ then $\alpha(v) = \gamma(h)$, v if $v \in V_A$ then

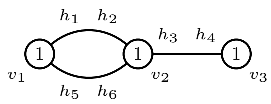
$$(\alpha^{-1}(v),\ \gamma^{-1}(v),\ \beta(a_A^{-1}(v)) \cup \zeta_A^{-1}(v),\ g_\Gamma, a_\Gamma, \iota_\Gamma, \zeta)$$

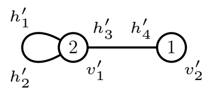
is a stable graph of genus $g(v)$ (where g_Γ, a_Γ and ι_Γ are restricted to the appropriate subsets and ζ is defined by ζ_Γ on $\zeta_A^{-1}(v)$ and by a_Γ on $\beta(a_A^{-1}(v))$).

Remark 2.1.6. If Γ has an A-structure (α, β, γ) and A has a B-structure $(\alpha', \beta', \gamma')$ it can be checked that $(\alpha' \circ \alpha, \beta \circ \beta', \gamma' \circ \gamma)$ is a B-structure on Γ. We can therefore define a *morphism of n-pointed genus g stable graphs* $\Gamma \to A$ as an A-structure on Γ. An *isomorphism* of stable graphs $A \to B$ is thus a B-structure (α, β, γ) on A and an A-structure $(\alpha', \beta', \gamma')$ on B such that $(\alpha' \circ \alpha, \beta \circ \beta', \gamma' \circ \gamma) = (\mathrm{id}_{V_A}, \mathrm{id}_{H_B}, \mathrm{id}_\emptyset)$ and $(\alpha \circ \alpha', \beta' \circ \beta, \gamma \circ \gamma') = (\mathrm{id}_{V_B}, \mathrm{id}_{H_A}, \mathrm{id}_\emptyset)$.

We can form an essentially finite category whose objects are stable genus g graphs with set of legs L and whose morphisms $\Gamma \to A$ are A-structures on Γ.

Example 2.1.7. Let

$\Gamma =$ 

, $A =$ 

One A-structure on Γ is

$$\begin{aligned} \alpha\colon v_1, v_2 &\mapsto v'_1 \\ v_3 &\mapsto v'_2 \\ \beta\colon h'_i &\mapsto h_i \\ \gamma\colon h_5, h_6 &\mapsto v'_1. \end{aligned}$$

In total there are 4 different A-structures which can be given to Γ. Indeed the choice of a map $\{h'_1\} \to \{h_1, h_2, h_5, h_6\}$ completely determines an A-structure on Γ.

Notation 2.1.8. For a stable graph Γ we define

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V} \overline{\mathcal{M}}_{g(v), n(v)}.$$

There is a natural *gluing morphism* $\xi_\Gamma\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ defined by Γ. It glues the curve

$$\left(\coprod_{v \in V} C_v \to B \ \ ; \ \ \sigma_h\colon B \to C_{a(h)},\ \sigma'_l\colon B \to C_{\zeta(l)},\ h \in H, l \in L \right) \in \overline{\mathcal{M}}_\Gamma$$

over B together by gluing the section σ_h to $\sigma_{\iota(h)}$. Note that this is a composition of gluing morphisms from Definition 1.3.1. The morphism ξ_Γ is a representable morphism of Deligne-Mumford stacks (see [ACG11, Proposition 10.25]) and since both its domain and codomain are smooth and complete by Theorem 1.1.29 the morphism ξ_Γ is an lci and proper morphism by Proposition 1.2.11.

For a given morphism of stable graphs $\Gamma \to A$ we define a corresponding map of moduli spaces

$$\xi_{\Gamma \to A}\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_A$$

as a composition of gluing morphisms.

Remark 2.1.9. The image of ξ_Γ in $\overline{\mathcal{M}}_{g,n}$ equals the closure of the locus of all curves in $\overline{\mathcal{M}}_{g,n}$ with dual graph isomorphic to Γ.

2.1.10. Let A and B be n-pointed genus g stable graphs. In this section we will compute the intersection

$$[\mathrm{Im}(\xi_A)] \cdot [\mathrm{Im}(\xi_B)]$$

as an explicit sum of classes of $\overline{\mathcal{M}}_{g,n}$. By the excess intersection formula (Proposition 1.2.13) we have to identify the fiber product

$$\begin{tikzcd}
\mathcal{F}_{A,B} \arrow[r, "p_2"] \arrow[d, "p_1"] & \overline{\mathcal{M}}_B \arrow[d, "\xi_B"] \\
\overline{\mathcal{M}}_A \arrow[r, "\xi_A"'] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

and the top Chern class of the excess intersection bundle $E = p_1^* N_{\xi_A}/N_{p_2}$. We will prove $\mathcal{F}_{A,B}$ is isomorphic to a disjoint union of stacks $\overline{\mathcal{M}}_\Gamma$ where Γ is a graph which appears as a specialization of both A and B.

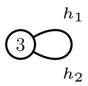
2.1.11. We will say that Γ has an (A,B)*-structure* (f,g), or that Γ *is an* (A,B)*-graph*, if Γ has both an A-structure $f = (\alpha_A, \beta_A, \gamma_A)$ and a B-structure $g = (\alpha_B, \beta_B, \gamma_B)$. We say that (f,g) is *a generic* (A,B)*-structure* on Γ if every edge of Γ corresponds to an edge of A or an edge of B, i.e. if

$$\beta_A(H_A) \cup \beta_B(H_B) = H_\Gamma.$$

We say that two stable graphs Γ, Γ' with (A,B)-structures (f,g) and (f',g') are *isomorphic* (A,B)*-graphs* if there exists an isomorphism $\tau : \Gamma \to \Gamma'$ such that the following diagram commutes

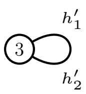
$$\begin{tikzcd}
 & B \\
\Gamma \arrow[ur, "g"] \arrow[r, "\tau"] \arrow[dr, "f"'] & \Gamma' \arrow[u, "g'"'] \arrow[d, "f'"] \\
 & A
\end{tikzcd} .$$

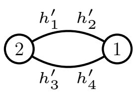
Example 2.1.12. Let

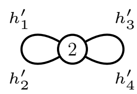
$A = B =$ 

.

There are three stable graphs which admit a generic (A,B)-structure:

$\Gamma_1 =$ 

$\Gamma_2 =$ 

$\Gamma_3 =$ 

The stable graph Γ_1 has two isomorphism classes of generic (A,B)-structures (f,g): Set $f = (\alpha_f, \beta_f, \gamma_f)$ and $g = (\alpha_g, \beta_g, \gamma_g)$. Up to isomorphism we can assume that $\beta_f(h_1) = h'_1$, there are then two possible nonisomorphic choices for $\beta_g(h_1)$, namely h'_1 and h'_2.

The graph Γ_2 has four isomorphism classes of generic (A,B)-structures: There is always an isomorphism such that β_f sends the edge (h_1, h_2) to (h'_1, h'_2). We then have $\beta_f(h_1) = h'_1$ or

$\beta_f(h_1) = h'_2$ and these choices are nonisomorphic. Since the (A, B)-structure is generic β_g must send (h_1, h_2) to (h'_3, h'_4) and we again have 2 choices.

The stable graph Γ_3 has only one isomorphism class of stable (A, B)-structures (f, g). Up to isomorphism we have $\beta_f(h_1) = h'_1$ and $\beta_g(h_1) = h'_3$.

Notation 2.1.13. Let A and B be stable n-pointed genus g graphs. We will denote by $\mathfrak{G}_{A,B}$ a set of representatives of the set of all isomorphism classes of generic (A, B)-graphs. We set

$$\mathcal{X} = \coprod_{(\Gamma,f,g)\in\mathfrak{G}_{A,B}} \overline{\mathcal{M}}_\Gamma.$$

Proposition 2.1.14. There is a natural isomorphism $\mathcal{X} \to \mathcal{F}_{A,B}$.

We start by giving a different modular interpretation of $\overline{\mathcal{M}}_\Gamma$ for any stable graph Γ (see [ACG11, page 315] or [GP03, Section A2]):

Definition 2.1.15. Let $\Gamma = (V, H, L, g, \iota, a, \zeta)$ be an n-pointed stable graph of genus g and let C be an n-pointed stable curve

$$\pi\colon C \to S, \qquad\qquad s_k\colon S \to C \quad k = 1, ..., n$$

of genus g over a connected base S. A Γ-*marking on* C is the following additional data: (this is called a Γ-structure in [GP03])

i $\#E$ additional sections $\sigma_1, ..., \sigma_{e(\Gamma)}$ of π with image in the singular locus of C,

ii $\#H$ sections $\tilde{\sigma}_{1,1}, \tilde{\sigma}_{1,2}, \tilde{\sigma}_{2,1}, ..., \tilde{\sigma}_{e(\Gamma),2}$ of the normalization $\tilde{C}$ of C along the sections σ_i,

iii $\#V$ disjoint connected components C_v of $C\backslash\{\sigma_i\}$ whose union is $C\backslash\{\sigma_i\}$ and such that each C_v remains connected upon pullback along any morphism $S' \to S$ of base schemes (we shall call such components π-*relative components of* $C\backslash\{\sigma_i\}$),

iv a choice of an isomorphism between Γ and the stable graph

$$(\{C_v\}, \{\tilde{\sigma}_{i,j}\}, \{s_k\}, g, \iota, a, \zeta)$$

where $g(C_v)$ is the arithmetic genus of C_v, the involution ι is defined by $\iota(\tilde{\sigma}_{i,1}) = \tilde{\sigma}_{i,2}$, a maps $\tilde{\sigma}_{i,j}$ to the π-relative component corresponding to the component of $\tilde{C}$ it lies on and ζ maps s_k to the π-relative component it lies on.

We will denote by C_Γ the curve C together with the data of a Γ-marking on C.

2.1.16. The data of a Γ-marking on a stable curve can be pulled back under any morphism of connected base schemes. We can therefore define a stack $\overline{\mathcal{M}}'_\Gamma$ whose objects consist of stable curves with Γ-marking and whose morphisms respect the Γ-marking under pullback.

Proposition 2.1.17. There exists a natural isomorphism between $\overline{\mathcal{M}}_\Gamma$ and $\overline{\mathcal{M}}'_\Gamma$.

Proof. We define a natural morphism from $\overline{\mathcal{M}}_\Gamma$ to $\overline{\mathcal{M}}'_\Gamma$ by assigning the canonical Γ-marking to the universal curve over $\overline{\mathcal{M}}_\Gamma$. In the other direction given a S-valued point C of $\overline{\mathcal{M}}'_\Gamma$ we naturally obtain a collection of $v(\Gamma)$ stable curves by analyzing the π-relative components of C. Since we have a bijection between these curves and $v(\Gamma)$ and a bijection between the sections of the normalization of C and the sections of families of curves in $\overline{\mathcal{M}}_\Gamma$, we obtain a S-valued point of $\overline{\mathcal{M}}_\Gamma$. It is straightforward to check that this correspondence induces a bijection on the collection of morphisms between the corresponding objects. □

Proof of Proposition 2.1.14. In this prove we will always use the modular interpretation of curves with a Γ-structure for $\overline{\mathcal{M}}_\Gamma$ and write $\overline{\mathcal{M}}_\Gamma$ for $\overline{\mathcal{M}}'_\Gamma$ everywhere.

Let $u\colon \mathcal{X} \to \overline{\mathcal{M}}_A$ be the map defined as $\xi_{f\colon \Gamma\to A}\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_A$ on the connected component $\overline{\mathcal{M}}_\Gamma$ of $\mathcal{X}$ indexed by (Γ, f, g). Similarly define $v\colon \mathcal{X} \to \overline{\mathcal{M}}_B$ to be the map $\xi_{g\colon \Gamma\to B}\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_B$ on the connected component of $\mathcal{X}$ indexed by (Γ, f, g). An object (C_Γ, f, g) of $\mathcal{X}$, over a connected base scheme S, consists of a graph Γ with a generic (A, B)-structure (f, g) together with a stable curve C over S endowed with a Γ-marking. Let (C_Γ, f, g) be one such object object of $\mathcal{X}$ over S. By definition we have $\xi_A(u(C_\Gamma, f, g)) = C = \xi_B(v(C_\Gamma, f, g))$ a natural isomorphism $\xi_A \circ u \Rightarrow \xi_B \circ v$ is therefore given by the identity. We have the following diagram:

$$\begin{tikzcd}
\mathcal{X} \arrow[dr, "q"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
& \mathcal{F}_{A,B} \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\Rightarrow"] & \overline{\mathcal{M}}_B \arrow[d, "\xi_B"] \\
& \overline{\mathcal{M}}_A \arrow[r, "\xi_A"'] & \overline{\mathcal{M}}_{g,n}.
\end{tikzcd}$$

where the map q is given by the strict universal property of the fiber product (see Remark 1.1.8). It sends the object (C_Γ, f, g) over S to the object $(C_A, C_B, \mathrm{id}_C)$ over S and a morphism $C'_\Gamma \to C_\Gamma$ over $S' \to S$ to the induced pair of morphisms $(C'_A \to C_A, C'_B \to C_B)$.

We want to prove that q is an isomorphism. We will do so by defining a map $r\colon \mathcal{F}_{A,B} \to \mathcal{X}$ and showing that $r \circ q$ and $q \circ r$ are naturally isomorphic to the respective identities on $\mathcal{X}$ and on $\mathcal{F}_{A,B}$.

Let $(D_A, C_B, \alpha\colon D \xrightarrow{\sim} C)$ be an object of $\mathcal{F}_{A,B}$ over S. Following the notation of Definition 2.1.15 we will define an A-marking on C by passing through the isomorphism α:

i if $\{\sigma_i\}$ are the $e(A)$ sections of $\varpi\colon D \to S$ in the singular locus of D defined by the A-marking, we get sections $\sigma'_i := \alpha \circ \sigma_i$ in the singular locus of C,

ii the pullback of α along the partial normalization $\tilde{C} \to C$ defines a map $\tilde{\alpha}\colon \tilde{D} \to \tilde{C}$, in this way we obtain sections $\{\tilde{\sigma}'_{i,j}\} := \{\tilde{\alpha} \circ \tilde{\sigma}_{i,j}\}$ in the partial normalization $\tilde{C}$ of C at $\{\sigma'_i\}$,

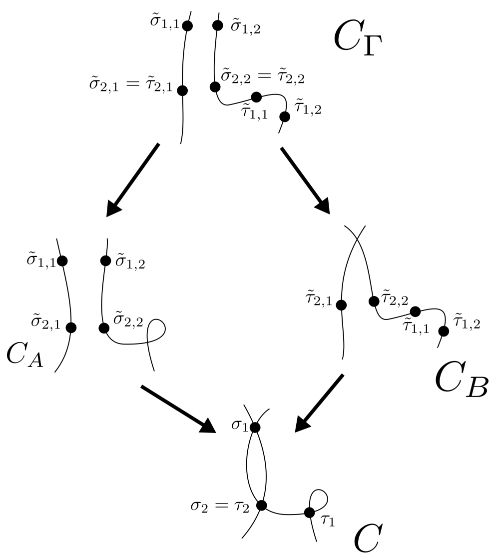
iii if $D_{v,A}$ are the ϖ' relative components of D then the π-relative components are given by $C_{v,A} := \alpha(D_{v,A})$,

iv we obtain an isomorphism of stable graphs by the composition

$$(\{C_{v,A}\}, \{\tilde{\sigma}'_{i,j}\}, \{\alpha \circ s_k\}, g', \iota', a', \zeta') \longrightarrow (\{D_{v,A}\}, \{\tilde{\sigma}_{i,j}\}, \{s_k\}, g, \iota, a, \zeta) \xrightarrow{\lambda} A$$

where λ is the isomorphism of stable graphs defined by the A-structure on D.

Let τ_i be the sections on C defined by the B-structure, $C_{v,B}$ the π-relative components defined by the B-structure. The curve C now comes with the following structure:



i a set of sections $E := \{\sigma'_i\} \cup \{\tau_i\}$ of π in the singular locus of C,

ii a set of sections $H := \{\tilde\sigma'_{i,j}\} \cup \{\tilde\tau_{i,j}\}$ in the partial normalization of C at E,

iii a set of π-relative components V of $C \backslash \{\sigma'_i\} \cup \{\tau_i\}$.

This data defines a stable graph

$$\Gamma := (V, H, \{\alpha(s_i)\}, g, \iota, a, \zeta)$$

as in Definition 2.1.15.iv (where the $\alpha(s_i)$ are the n sections of C outside of the singular locus corresponding to the marked points). This data defines a Γ-marking on C.

This Γ has an (A,B)-structure, indeed let $\alpha_A\colon V \to V_A$, be the map of π-relative components given by the inclusion $C\backslash\{\sigma'_i\} \cup \{\tau_i\} \hookrightarrow C\backslash\{\sigma'_i\}$, let $\beta_A\colon H_A \hookrightarrow H$ be the obvious inclusion of sections and let $\gamma_A\colon H\backslash \operatorname{Im}\beta \to H_A$ be the map that sends $\tilde\tau_{i,j}$ to the π-relative component $C_{v,A}$ in which τ_i lies. In this way we have constructed an A-structure $f = (\alpha_A, \beta_A, \gamma_A)$ on Γ and we can define a B-structure g on Γ in the same way. In other words we have defined an object (C_Γ, f, g) of $\mathcal{X}$ over S. This completes the definition of the functor r on the objects of $\mathcal{F}_{A,B}$.

Let $(\lambda_1\colon D'_A \to D_A, \lambda_2\colon C'_B \to C_B)$ be a morphism in $\mathcal{F}_{A,B}$ over $\lambda\colon S' \to S$. Let $C'_{\Gamma'}$ and C_Γ be the curves with Γ' and Γ-structure defined as above by D'_A, C'_B and respectively D_A, C_B. The maps λ_1 and λ_2 together define an isomorphism of stable graph $\Gamma' \to \Gamma$ which commutes with the respective (A,B)-structures (f', g') and (f,g) of these graphs. In other words this defines an isomorphism of (A,B)-graphs and a map

$$(C'_{\Gamma'}, f', g') \to (C_\Gamma, f, g).$$

This completes the definition of the functor r on the morphisms of $\mathcal{F}_{A,B}$.

It remains to check that r and q are inverses of each other. Let (C_Γ, f, g) be an object of $\mathcal{X}$. Then $q(C_\Gamma, f, g) = (C_A, C_B, \mathrm{id}_C)$. If we pass this through the above construction we see that $r = (C_A, C_B, \mathrm{id}_C) = (C_{\Gamma'}, f', g')$ where Γ and Γ' are isomorphic (A,B)-graphs. In other words $r \circ q$ is the identity on $\mathcal{X}$.

In the other direction let $(D_A, C_B, \alpha\colon D \to C)$ be a S-point of $\mathcal{F}_{A,B}$. We have

$$q(r(D_A, C_B, \alpha)) = q((C_\Gamma, f, g)) = (C_A, C_B, \alpha).$$

An isomorphism between D_A and C_A is given by passing the A-structure through α as above when we define the functor r. It is clear that this defines an isomorphism of objects $(D_A, C_B, \alpha) \to (C_A, C_B, \alpha)$. It follows that $q \circ r$ is naturally isomorphic to the identity on $\mathcal{F}_{A,B}$. □

2.1.18. Let A and B be n-pointed genus g stable graphs. We will now identify the excess bundle $E := p_1^* N_{\xi_A}/N_{p_2}$ on $\mathcal{F}_{A,B}$ of the intersection between $\overline{\mathcal{M}}_A$ and $\overline{\mathcal{M}}_B$ (see Paragraph 2.1.10). A vector bundle on $\mathcal{F}_{A,B}$ is the same as a vector bundle on each of its connected components.

For any $(\Gamma, f, g) \in \mathfrak{G}_{A,B}$ therefore consider the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_\Gamma & \xrightarrow{\xi_{\Gamma\to B}} & \overline{\mathcal{M}}_B \\ {\scriptstyle \xi_{\Gamma\to A}}\downarrow & & \downarrow{\scriptstyle \xi_B} \\ \overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,n} \end{array} .$$

We will identify the vector bundle

$$E_\Gamma := \xi_{\Gamma\to A}^* N_{\xi_A}/N_{\xi_{\Gamma\to B}}.$$

and then compute its top Chern class $c_{\mathrm{top}}(E_\Gamma)$.

Recall (see for example [ACG11, Section 13.3]) that the normal bundle N_{ξ_A} can be identified with

$$N_{\xi_A} = \bigoplus_{(h,h')\in E_A} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee,$$

where $\mathbb{L}_h$ is defined as in Definition 1.3.4. Similarly

$$N_{\xi_{\Gamma\to B}} = \bigoplus_{(h,h')\in E_B \setminus \operatorname{Im}\beta_B} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee.$$

It follows that

$$E_\Gamma = \bigoplus_{(h,h')\in \operatorname{Im}\beta_A \cap \operatorname{Im}\beta_B} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee.$$

Since $c_1(\mathbb{L}_h) = \psi_h$, the top Chern class of E is the product over the relevant ψ-classes.

In conclusion we get:

Proposition 2.1.19. Let A and B be stable graphs, then

$$\xi_A^* \xi_{B*}([\overline{\mathcal{M}}_B]) = \sum_{\Gamma\in\mathfrak{G}_{A,B}} \xi_{\Gamma\to A*}\left(\prod_{(h,h')\in \operatorname{Im}\beta_A\cap\operatorname{Im}\beta_B} (-\psi_h - \psi_{h'}) \cdot [\overline{\mathcal{M}}_\Gamma]\right)$$

Proof. This now follows directly from Corollary 1.2.14, Proposition 2.1.14 and Paragraph 2.1.18. □

Decorated Boundary Strata

We will now define decorated stratum classes, compute the product of two decorated stratum classes in terms of decorated stratum classes and deduce some useful consequences.

2.1.20. Let A be a stable graph and $v \in V$. Consider the projection map

$$p_v \colon \overline{\mathcal{M}}_A = \prod_{w \in V} \overline{\mathcal{M}}_{g(w),n(w)} \to \overline{\mathcal{M}}_{g(v),n(v)}.$$

We will set $\kappa_{v,i} := p_v^*(\kappa_i) \in A^i(\overline{\mathcal{M}}_A)$ where κ_i is the class from Definition 1.3.4 and $\psi_{v,i} := p_v^*(\psi_i) \in A^1(\overline{\mathcal{M}}_A)$.

Definition 2.1.21. A *decorated stable graph* A_θ is a stable graph A together with a *decoration*

$$\theta = \prod_{v \in V} \left(\prod_{i \in a^{-1}(v) \cup \zeta^{-1}(v)} \psi_{v,i}^{a_i} \prod_{j=1}^{m} \kappa_{v,j}^{b_j} \right) \in A^\bullet(\overline{\mathcal{M}}_A).$$

Remark 2.1.22. The decoration θ restricted to each vertex is just a monomial in ψ and κ classes. Given another decoration θ', the product $\theta \cdot \theta'$ in $A^\bullet(\overline{\mathcal{M}}_A)$ is defined by

$$\theta \cdot \theta' = \prod_{v \in V} \left(\prod_{i \in a^{-1}(v) \cup \zeta^{-1}(v)} \psi_{v,i}^{a_i + a_i'} \prod_{j=1}^{m} \kappa_{v,j}^{b_j + b_j'} \right)$$

Remark 2.1.23. A decorated stable graph A_θ naturally defines a bivariant class $\theta \cap \xi_A^*(\_) \in A^\bullet(\overline{\mathcal{M}}_A \xrightarrow{\xi_A} \overline{\mathcal{M}}_{g,n})$.

Notation 2.1.24. We set

$$[A_\theta] := \frac{1}{|\operatorname{Aut} A|} \xi_{A*}(\theta) \in A^\bullet(\overline{\mathcal{M}}_{g,n})$$

and will call the class $[A_\theta]$ a *decorated stratum class*. We will drop θ from the notation if $\theta = 1$, note that using this notation $[A] = [\operatorname{Im}(\xi_A)]$. We will draw decorated stratum classes by attaching a number of arrowheads to each half edge or leg i equal to a_i and by attaching the monomial

$$\prod_{j=1}^{m} \kappa_{v,j}^{b_j}$$

to each vertex v.

Example 2.1.25. Let A be the graph $\textcircled{2}\!-\!\textcircled{3}$. If v_1 is the vertex of genus 2 and v_2 is the vertex of genus 3 and we have a decoration $\theta = \psi_{v_2,h}^2 \kappa_{v_1,1}^2$ we will draw the decorated graph A_θ as

$$\overset{\kappa_1^2}{\textcircled{2}} \twoheadrightarrow \textcircled{3}.$$

Warning 2.1.26. There are two conflicting notations in the literature for decorated stratum classes. One is as given in 2.1.24 the other one is without dividing by the size of the automorphism group of A.

The advantage of our definition is that the class $[A]$ is the Poincaré dual of the closure of the locus of all stable curves with dual graph isomorphic to A. In particular $[A]$ corresponds to an actual closed integral substack of $\overline{\mathcal{M}}_{g,n}$. The advantage of not dividing by the order of the automorphism group is that in practice it makes calculating various intersections easier.

Remark 2.1.27. The codimension (or degree) of ψ_i in $A^\bullet(\overline{\mathcal{M}}_{g,n})$ is 1, the codimension of κ_j is j and the codimension of $[A] \in A^\bullet(\overline{\mathcal{M}}_{g,n})$ equals the number of nodes of A. Therefore if A_θ is a decorated boundary graph with decoration

$$\theta = \prod_{v \in V} \left(\prod_{i \in a^{-1}(v) \cup \zeta^{-1}(v)} \psi_{v,i}^{a_i} \prod_{j=1}^{m} \kappa_{v,j}^{b_j} \right)$$

then

$$\operatorname{codim}[A_\theta] = \#E_A + \sum_{i \in H \cup L} a_i + \sum_{v \in V, j} j b_{v,j}.$$

Let A_θ and B_λ be decorated stratum classes, we will now determine the product $[A_\theta] \cdot [B_\lambda]$ as a sum of decorated stratum classes. For this we need to know the pullback of θ under $\xi_{\Gamma \to A}$. Since the pullback is a ring homomorphism, this can be done by pulling back each ψ and κ class in θ.

Lemma 2.1.28. Let $f = (\alpha, \beta, \gamma) \colon \Gamma \to A$ be a map of stable graphs, then

$$\xi^*_{f \colon \Gamma \to A}(\psi_{v,i}) = \psi_{a_\Gamma \circ \beta(i), \beta(i)},$$
$$\xi^*_{f \colon \Gamma \to A}(\kappa_{v,i}) = \sum_{w \in \alpha^{-1}(v)} \kappa_{w,i}.$$

Proof. The first of these is trivial, the second is [ACG11, Lemma 4.31]. □

Corollary 2.1.29. Let A and Γ be stable graphs, let f be an A-structure on Γ and let θ be a decoration on A. We have

$$\xi_f^*(\theta) = \prod_{v \in V_A} \left(\prod_{i=1}^{n(v)} \psi^{a_i}_{a_\Gamma \circ \beta_f(i), \beta_f(i)} \prod_{j=1}^{m} \left(\sum_{w \in \alpha_f^{-1}(v)} \kappa_{w,j} \right)^{b_j} \right)$$

Theorem 2.1.30. Let A_θ and B_λ be decorated stable n-pointed genus g graphs. Then

$$[A_\theta] \cdot [B_\lambda] =$$
$$\frac{1}{|\operatorname{Aut} A| \cdot |\operatorname{Aut} B|} \sum_{(\Gamma, f, g) \in \mathfrak{G}_{A,B}} \xi_{\Gamma *} \left(\xi_f^*(\theta) \cdot \xi_g^*(\lambda) \prod_{(h,h') \in \operatorname{Im} \beta_A \cap \operatorname{Im} \beta_B} (-\psi_h - \psi_{h'}) \right).$$

Proof. This follows by pushing forward the expression of Proposition 2.1.19. □

Proposition 2.1.31. Let $[A_\theta]$ be a decorated $n+1$ pointed genus g boundary graph. The pushforward of $[A_\theta]$ under the forgetful morphism $\pi \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is a decorated stratum class.

Proof. Let l be the leg forgotten corresponding to the last point. If l is attached to a vertex v of A of genus 0 with $n(v) = 3$ then this vertex cannot be nonzero and decorated by any ψ or κ classes (otherwise $[A_\theta] = 0$). Let B be the stable graph given by contracting the vertex v of A. In this case $\pi_*([A_\theta]) = [B_\theta]$.

If v is not a vertex of genus 0 with $n(v) = 0$, then $\pi_*([A_\theta])$ is given by pushing forward the κ and ψ classes on v through the forgetful morphism (see Example 1.3.8). □

Corollary 2.1.32. The tautological ring $R^\bullet(\overline{\mathcal{M}}_{g,n})$ is additively generated by the decorated stratum classes.

Proof. Recall that ψ and κ lie in the tautological ring (see Proposition 1.3.5). It follows that decorated stratum classes lie in the tautological ring since they are the pushforward of a product of ψ and κ classes under a collection of gluing morphisms.

The subgroup of $R^\bullet(\overline{\mathcal{M}}_{g,n})$ additively generated by decorated stratum classes is closed under products since the product between two decorated stratum classes is a sum of decorated stratum classes by Theorem 2.1.30.

The pushforward of a decorated stratum class under the forgetful morphism is a decorated stratum class by Proposition 2.1.31. □

Proposition 2.1.33 ([GP03, Proposition 1])**.** Let A be a stable n-pointed genus g graph and let $\gamma \in R^\bullet(\overline{\mathcal{M}}_{g,n})$. Then

$$\xi_A^*(\gamma) \in \bigotimes_{v \in V_A} R^\bullet(\overline{\mathcal{M}}_{g(v),n(v)}).$$

where ξ_A is the gluing morphism defined in Notation 2.1.8.

Proof. By Corollary 2.1.32 we can take γ to be a decorated stratum class. In analogy with Theorem 2.1.30 we see that the pullback of a decorated stratum class $[B_\theta]$ under ξ_A decomposes as

$$\frac{1}{|\operatorname{Aut} B|} \sum_{(\Gamma,f,g) \in \mathfrak{G}_{A,B}} \xi_{f*}\left(\xi_g^*(\theta) \prod_{(h,h') \in \operatorname{Im}\beta_A \cap \operatorname{Im}\beta_B} (-\psi_h - \psi_{h'})\right)$$

which yields the desired decomposition. □

Definition 2.1.34. We say that a cycle $\lambda \in \bigotimes_i H^\bullet(\overline{\mathcal{M}}_{g_i,n_i})$ *admits a tautological Künneth decomposition* if $\lambda \in \bigotimes_i RH^\bullet(\overline{\mathcal{M}}_{g_i,n_i})$.

Remark 2.1.35. Proposition 2.1.33 implies that the pullback along a gluing morphism of a tautological class admits a tautological Künneth decomposition.

Remark 2.1.36. Theorem 2.1.19 was originally proven by [GP03]. Their main purpose was to prove the existence of a Künneth decomposition for tautological classes and they only cared about the existence of such a formula.

Computer programs implementing Theorem 2.1.30 have been written by Stephanie Yang [Yan08] using maple and by Aaron Pixton [Pix] using sage. Pixton's program is also able to compute a list of all the extended Faber-Zagier relations efficiently for low g and n (see 1.3.6). Johannes Schmitt has further extended Pixton's program to be able to compute the decomposition of Proposition 2.1.33 and the pushforward of sums of decorated stratum classes under forgetful maps (and more). Schmitt's program [Sch] is used in some of these calculations of the later sections. A paper detailing its use is forthcoming.

2.2 Intersections with Admissible Covers

In this section we will compute the intersection of a space of admissible (double) covers with a decorated stratum class. Let $\phi_n\colon \mathcal{A}dm(g,h)_{2m} \to \overline{\mathcal{M}}_{g,n+2m}$ be the source map defined in Notation 1.3.3 and let $\rho\colon \mathcal{A}dm(g,h)_{2m} \to \overline{\mathcal{M}}_{h,2g+2-4h+m}$ be the target map. Let A_θ be a decorated stratum class of $\overline{\mathcal{M}}_{g,n+2m}$. In this chapter we will compute the pullback $\xi_A^* \phi_{n*}[\mathcal{A}dm(g,h)_{2m}]$ in terms of spaces of admissible covers and ψ-classes in Theorem 2.2.21 and the pull-push $\rho_* \phi_n^*([A_\theta])$ as a sum of decorated stratum classes on $\overline{\mathcal{M}}_{h,2g+2-4h+m}$ in Theorem 2.2.26.

Specifically, we will give a combinatorial description of the fiber product

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta"] \arrow[d, "\phi'_n"] & \mathcal{A}dm(g,h)_{2m} \arrow[d, "\phi_n"] \\ \overline{\mathcal{M}}_A \arrow[r, "\xi_A"] & \overline{\mathcal{M}}_{g,n+2m} \end{tikzcd},$$

and identify its excess bundle $E = \phi_n'^* N_{\xi_A}/N_\eta$. We will then also give the pullback ${\phi'_n}^*(\theta)$ and compute the pushforward

$$\rho_* \eta_* ({\phi'_n}^*(\theta) \cdot c_{\mathrm{top}}(E) \cap [\mathcal{F}])$$

as a sum of decorated stratum classes.

We will start by reducing everything to the case where we do not forget any of the points, i.e. where $\phi = \phi_n$ (Notation 1.3.3). In other words we will start by reviewing the known results about the pullback of a decorated stratum class $[A_\theta]$ under the forgetful morphism.

2.2.1. Let A be a stable n-pointed graph and consider the fiber product

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n+k} \arrow[d, "\pi^{(k)}"] \\ \overline{\mathcal{M}}_A \arrow[r, "\xi_A"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

where $\pi^{(k)}$ is the forgetful morphism which forgets the last k points and stabilizes. Let $\mathfrak{A}$ be the set of all graphs

$$(V, H, [n+k], g, \iota, a, \zeta)$$

such that

$$(V, H, [n], g, \iota, a, \zeta|_{[n]}) = A.$$

Set $\mathcal{Y} := \coprod_{B \in \mathfrak{A}} \overline{\mathcal{M}}_B$. There is a natural map $\tilde{\pi}^{(k)} \colon \mathcal{Y} \to \overline{\mathcal{M}}_A$ given by forgetting the last k points and stabilizing and a natural map $\xi_{\mathfrak{A}} \colon \mathcal{Y} \to \overline{\mathcal{M}}_{g,n+k}$ given by gluing. We therefore get a map $\chi \colon \mathcal{Y} \to \mathcal{F}$, this map is not an isomorphism, indeed $\mathcal{F}$ is connected while $\mathcal{Y}$ is not. But χ is proper and generically 1-to-1 (unwrapping definitions we see that χ is surjective, and $\xi_{\mathfrak{A}}$ is generically injective so χ is as well). Therefore $\chi_*[\mathcal{Y}] = \mathcal{F}$.

Let $\pi_B \colon \overline{\mathcal{M}}_B \to \overline{\mathcal{M}}_A$ be the restriction of the forgetful map to $\overline{\mathcal{M}}_B$, then $\pi_B^*(\psi_i)$ and $\pi_B^*(\kappa_a)$ can be determined by using the following two lemmas.

Lemma 2.2.2 (Extended comparison result)**.** Let $\pi^{(k)} \colon \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism and let $\psi_i \in A^\bullet(\overline{\mathcal{M}}_{g,n})$ and $\tilde{\psi_i} \in A^\bullet(\overline{\mathcal{M}}_{g,n+k})$ be the respective ψ classes. Then

$$\tilde{\psi_i} = \pi^{(k)*}(\psi_i) + D$$

where D is the sum of all divisors which have the marking P_i on a component R of genus 0 and all other markings P_j, $1 \leq j \leq n$ on a component of genus g not containing R.

Proof. This result is follows from repeated application of [ACG11, Lemma 17.4.28.ii]. □

Lemma 2.2.3. Let $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. We have $\pi^*(\kappa_a) = \kappa_a - \psi_{n+1}^a$.

Proof. This is [ACG11, Lemma 17.4.28.i]. □

Remark 2.2.4. Let $[A_\theta]$ be a decorated stratum class and $\pi^{(k)}\colon \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$. The discussion of Paragraph 2.2.1, the extended comparison result and repeated application of Lemma 2.2.3 now completely determine the pullback $\pi^{(k)*}([A_\theta])$ as a sum of decorated stratum classes on $\overline{\mathcal{M}}_{g,n+k}$.

Corollary 2.2.5. The tautological ring is closed under pullback along gluing and forgetful morphisms.

Proof. This follows immediately by the Künneth decomposition of Proposition 2.1.33 and by Remark 2.2.4. □

The Fiber Product

Let A be a $2g+2-4h+2m$ pointed graph of genus g. In this subsection we will study the fiber product

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{A}dm(g,h)_{2m} \arrow[d, "\phi"] \\ \overline{\mathcal{M}}_A \arrow[r, "\xi_A"] & \overline{\mathcal{M}}_{g,2g+2-4h+2m} \end{tikzcd}.$$

We will start by defining a notion of an involution τ on a stable graph Γ. We will do this in such a way that we can form a quotient stable graph of Γ under τ. We will also ensure that this definition corresponds to the action of the covering involution of an admissible cover $S \to T$ on the dual graph of S. In Definition 2.1.5 we gave the notion of a morphism (and hence of an isomorphism and automorphism) of stable graphs. This definition was the right one for the purposes of Section 2.1 but it does not define the right notion of an involution for our current purposes for the following reasons:

- an automorphism of stable graphs $\Gamma \to \Gamma$ fixes the set of legs of Γ. We will want our involution to permute some of the legs, in analogy with what happens for the $2m$ points on admissible double covers,
- the quotient of the set of vertices, half edges and legs of a graph Γ under the action of an automorphism of Γ does not necessarily define a stable graph,
- there are many possible choices for a genus function for the quotient of a stable graph Γ under an automorphism τ. The correct choice is the one given by the Riemann-Hurwitz theorem.

With these issues in mind we now introduce the following rather technical definition.

Definition 2.2.6. Assume we are given the initial data of three nonnegative integers g, h and m, of a set L of cardinality $2g+2-4h+2m$ and of an involution $\sigma \in \operatorname{Aut} L$ fixing $2g+2-4h$ elements. Then let $\Gamma = (V, H, L, g, \iota, a, \zeta)$ be a stable graph of genus g and let $\tau = (\alpha, \beta, \gamma)$ be an isomorphism of stable graphs

$$\tau\colon (V, H, L, g, \iota, a, \zeta) \to (V, H, L, g, \iota, a, \zeta \circ \sigma).$$

such that α and β are involutions.

Since β commutes with ι, the involution ι induces a well defined involution ι_τ on H/β. Let $v \in V$ and $h \in H$, by definition $a(h) = v$ if and only if $a(\beta(h)) = \alpha(v)$, therefore there is a well defined function $a_\tau \colon H/\beta \to V/\alpha$. Similarly if $l \in L$ and $v \in V$ then $\zeta(l) = v$ if and only if $\zeta(\sigma(l)) = \alpha(v)$ so there is a well defined function $\zeta_\tau \colon L/\sigma \to V/\alpha$.

For a vertex $v \in V$ we will denote by $r(v)$ the number of half edges and legs incident to v fixed by τ. Let $h \colon V/\alpha \to \mathbb{Q}$ be the map that sends $\mathrm{Orb}_\alpha v \in V/\alpha$ to $\frac{1}{4}(2g(v) + 2 - r(v))$ if $\mathrm{Orb}\, v$ consists of one element and to $g(v) = g(\alpha(v))$ if $\mathrm{Orb}_\alpha v = \{v, \alpha(v)\}$ consists of two distinct elements.

We say that (Γ, τ) is an *admissible pair* if the septuple

$$\Gamma/\tau := (V/\alpha, H/\beta, L/\sigma, h, \iota_\tau, a_\tau, \zeta_\tau)$$

is a stable graph. This is equivalent to requiring that:

i the image of h is contained in $\mathbb{Z}_{\geq 0}$,

ii ι_τ has no fixed points.

Remark 2.2.7. Let (Γ, τ) be an admissible pair. By induction over the set of edges it follows that $g(\Gamma/\tau) = h$.

Remark 2.2.8. We will always choose the involution σ of Definition 2.2.6 to be consistent with the map $\phi \colon \mathcal{A}dm(g,h)_{2m} \to \overline{\mathcal{M}}_{g,2g+2-4h+2m}$. Specifically if L is the set of marked points for source curves in $\mathcal{A}dm(g,h)_{2m}$ we will let $\sigma \in \mathrm{Aut}\, L$ be the involution fixing the ramification points of the covering involution and pairwise permuting the remaining $2m$ points in accordance with the covering involution.

Definition 2.2.9. Let (Γ, τ) be an admissible pair and A a stable graph. We define an *A-structure f on* (Γ, τ) to be an A-structure $f = (\alpha, \beta, \gamma)$ on Γ. We say that the A-structure on (Γ, τ) is *generic* if

$$\mathrm{Im}\, \beta \cup \tau(\mathrm{Im}\, \beta) = H_\Gamma.$$

If A is a stable graph we say that A *admits an admissible pair of genus h* if there exists an admissible pair (Γ, τ) with a generic A-structure and $g(\Gamma/\tau) = h$. We will say that A *admits an hyperelliptic or bielliptic pair* if it admits an admissible pair of genus $h = 0$ or genus $h = 1$ respectively.

Definition 2.2.10. An *isomorphism of admissible pairs* $(\Gamma', \tau') \to (\Gamma, \tau)$ is an isomorphism of stable graphs $g \colon \Gamma' \to \Gamma$ such that $\tau \circ g = g \circ \tau'$.

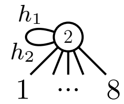
An *isomorphism of A structures* $(\Gamma', \tau', f') \to (\Gamma, \tau, f)$ is an isomorphism of admissible pairs $(\Gamma', \tau') \to (\Gamma, \tau)$ such that the induced diagram of stable graphs

$$\begin{tikzcd} & A \\ \Gamma' \arrow[ur, "f'"] \arrow[r] & \Gamma \arrow[u, "f"'] \end{tikzcd}$$

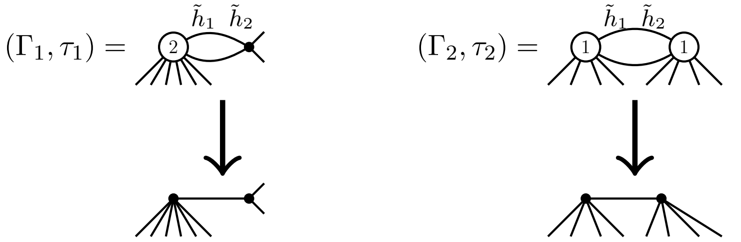
commutes.

Notation 2.2.11. Let g, h, m be non-negative, let L be a set of cardinality $2g+2-4h+2m$ and $\sigma \in \operatorname{Aut} L$ be an involution fixing $2g+2-4h$ elements. Let A be a genus g graph with set of legs L. We will denote by $\mathfrak{H}_A$ a set of representatives of isomorphism classes of generic A-structures (Γ, τ, f) with genus $g(\Gamma/\tau) = h$.

Example 2.2.12. Let

$A =$ 

The stable graph A admits admissible hyperelliptic pairs of the following form:



where τ_i is the involution fixing all the legs and all the vertices and switching the edges. Indeed for any hyperelliptic pair (Γ, τ) the graph Γ/τ cannot have any loops. This means that if B is a graph admitting an hyperelliptic pair (Γ, τ) with a loop $e = (h_1, h_2)$ then the image of e in Γ cannot be fixed by τ. If B has no further edges then the condition that the B-structure on Γ is generic means that Γ cannot have more than 2 edges. In our case the possible admissible hyperelliptic pairs are now completely determined by the condition that for each vertex v the involution τ must fix exactly $2g(v)+2$ legs or half edges incident to v.

There are in total $\binom{8}{6}$ hyperelliptic pairs of the form (Γ_1, τ_1) given by the different distributions of the legs. Likewise there are $\binom{8}{4}$ hyperelliptic pairs of the form (Γ_2, τ_2).

For each A-structure $f = (\alpha, \beta, \gamma)$ on (Γ_1, τ_1) there exists an isomorphism of A-structures such that the edge (h_1, h_2) is mapped to $(\tilde{h}_1, \tilde{h}_2)$. There are two possible choices either $\beta(h_1) = \tilde{h}_1$ or $\beta(h_1) = \tilde{h}_2$. After these choices the A-structure f is completely determined so there are 2 isomorphism classes of generic A-structure on (Γ_1, τ_1). The same argument holds for (Γ_2, τ_2) and there are 2 isomorphism classes of A-structures on (Γ_2, τ_2).

Notation 2.2.13. Let (Γ, τ) be an admissible pair with h the genus function of the quotient graph Γ/τ. Let $V^\tau \subset V/\tau$ be the set of trivial orbits of V under τ and let V^c the set of nontrivial orbits. Denote by $m(v)$ the number of half edges and legs of a vertex $v \in V$ switched by the involution (in other words $m(v) = n(v) - (2g(v) + 2 - 4h(\operatorname{Orb} v))$). We define a stack

$$\mathcal{Adm}_{(\Gamma,\tau)} := \prod_{v \in V^\tau} \mathcal{Adm}(g(v), h(v))_{2m(v)} \times \prod_{v \in V^c} \overline{\mathcal{M}}^D_{h(v),n(v)}.$$

There is a map

$$\eta_\Gamma \colon \mathcal{Adm}_{(\Gamma,\tau)} \to \mathcal{Adm}(g,h)_{2m}$$

which glues together the source and the target curves in accordance to Γ and Γ/τ respectively, as in Notation 2.1.8. This defines a representable lci morphism.

There is also a map

$$\phi_\Gamma \colon \mathcal{A}dm_{(\Gamma,\tau)} \to \overline{\mathcal{M}}_\Gamma$$

given by taking the data of the source curve and forgetting the admissible maps and target curves everywhere. If (Γ,τ) has an A-structure we can define a map $\phi_f \colon \mathcal{A}dm_{(\Gamma,\tau)} \to \overline{\mathcal{M}}_A$ by the composition $\xi_{f\colon \Gamma\to A} \circ \phi_\Gamma \colon \mathcal{A}dm_{(\Gamma,\tau)} \to \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_A$.

Definition 2.2.14. Let $S \to T$ be an admissible double cover over a connected base scheme B with involution $\tau' \colon S \to S$ induced by the admissible cover. A (Γ,τ)-marking on $S \to T$ is a Γ-marking on S (see Definition 2.1.15) such that the involution τ is induced by τ'.

2.2.15. A (Γ,τ)-marking on $S \to T$ defines a Γ-marking on S and a Γ/τ-marking on T. The data of a (Γ,τ)-structure can be pulled back along morphisms of base schemes $\Gamma' \to \Gamma$ (by definition of the pullback of an admissible cover along such a morphism and the pullback of the Γ-structure on S). We define a stack $\mathcal{A}dm'_{(\Gamma,\tau)}$ of admissible covers with a (Γ,τ)-marking.

Proposition 2.2.16. There is a natural isomorphism $\mathcal{A}dm'_{(\Gamma,\tau)} \to \mathcal{A}dm_{(\Gamma,\tau)}$.

Proof. The proof is completely analogous to that of Proposition 2.1.17. □

2.2.17. Let A be a stable graph and consider the space

$$\mathcal{X} = \coprod_{(\Gamma,\tau,f)\in\mathfrak{H}_A} \mathcal{A}dm_{(\Gamma,\tau)}\,.$$

We define a map $\phi_{\mathfrak{H}} \colon \mathcal{X} \to \overline{\mathcal{M}}_A$ by taking the map $\phi_{f\colon \Gamma\to A} \colon \mathcal{A}dm_{(\Gamma,\tau)} \to \overline{\mathcal{M}}_A$ on each irreducible component of $\mathcal{X}$. We define a map $\eta_{\mathfrak{H}} \colon \mathcal{X} \to \mathcal{A}dm(g,h)_{2m}$ by the map η_Γ on each irreducible component of $\mathcal{X}$.

Proposition 2.2.18. Let $\mathcal{F} := \overline{\mathcal{M}}_A \times_{\overline{\mathcal{M}}_{g,2g+2-4h+2m}} \mathcal{A}dm(g,h)_{2m}$ be the fiber product, with the above notation there exists an isomorphism of stacks $\mathcal{F} \simeq \mathcal{X}$.

The proof is very similar to that of Proposition 2.1.14.

Proof. We use the modular interpretation of $\mathcal{A}dm_{(\Gamma,\tau)}$ given by admissible covers with a (Γ,τ)-marking (see Definition 2.2.14 and Proposition 2.2.16) and the modular interpretation of $\overline{\mathcal{M}}_A$ in terms of curves with an A-marking (Definition 2.1.15).

An object $(S_{(\Gamma,\tau)} \to T_{\Gamma/\tau}, f)$ of $\mathcal{X}$ over B consists of an admissible pair (Γ,τ) together with a generic A-structure f and an admissible map $S \to T$ with a (Γ,τ)-marking. By definition we have $\xi_A \circ \phi_{\mathfrak{H}}(S_{(\Gamma,\tau)} \to T_{\Gamma/\tau}, f) = S = \phi \circ \eta_{\mathfrak{H}}(S_{(\Gamma,\tau)} \to T_{\Gamma/\tau}, f)$. We have a natural isomorphism $\xi_A \circ \phi_{\mathfrak{H}} \Rightarrow \phi \circ \eta_{\mathfrak{H}}$ given by the identity, so the strict universal property of fiber products gives us a morphism $q \colon \mathcal{X} \to \mathcal{F}_{A,\mathcal{H}}$. In other words we have a diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[dr, "q"] \arrow[drr, bend left, "\eta_{\mathfrak{H}}"] \arrow[ddr, bend right, "\phi_{\mathfrak{H}}"'] & & \\
& \mathcal{F} \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{A}dm(g,h)_{2m} \arrow[d, "\phi"] \\
& \overline{\mathcal{M}}_A \arrow[r, "\xi_A"'] & \overline{\mathcal{M}}_{g,2g+2+2m}.
\end{tikzcd}$$

We will construct an inverse r to q.

Let $(C_A, S \to T, \alpha\colon C \xrightarrow{\sim} S)$ be an object of $\mathcal{F}$ over B. As explained in the proof of Proposition 2.1.14 the A-structure on C again passes through α to define an A-structure on S. Let τ be the covering involution of $S \to T$. We now have the following data on this admissible cover

i a set of sections $E := \{\sigma_i\}_i \cup \{\tau(\sigma_i)\}_i$ in the singular locus of S where σ_i are the sections defined by the A-structure on S,

ii a set of sections $H := \{\tilde{\sigma}_{i,j}\}_{i,j} \cup \{\tau(\tilde{\sigma}_{i,j})\}_{i,j}$ in the normalization $\tilde{S}$ of S,

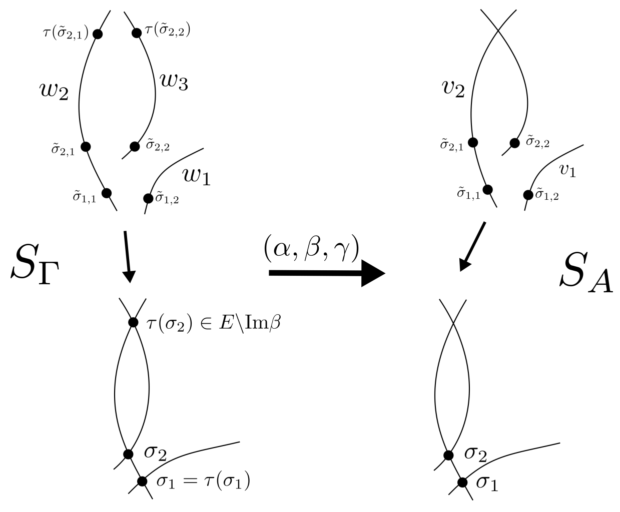
iii a set V of π-relative components of $S \backslash (\{\sigma_i\} \cup \{\tau(\sigma_i)\})$

This data defines a stable graph

$$\Gamma := (V, H, L, g, \iota, a, \zeta)$$

where g is the genus function that assigns the arithmetic genus to each element of V, ι is the involution defined by $\iota(\tilde{\sigma}_{i,1}) = \tilde{\sigma}_{i,2}$ and $\iota(\tau(\tilde{\sigma}_{i,1})) = \tau(\tilde{\sigma}_{i,2})$ and $a\colon H \to V$ is the function that sends a section to the π-relative component it lies on. The involution τ has an induced action on Γ and (Γ, τ) is an admissible pair.

Let $\beta\colon H_A \hookrightarrow H$ be the obvious inclusion. Let $\alpha\colon V \to V_A$ be the map sending a π relative component of S_Γ to the π relative component of S_A it is mapped to under the partial normalization defined by the sections $E \backslash \operatorname{Im} \beta$. Let $\gamma\colon H \backslash \operatorname{Im} \beta \to V_A$ be the map which sends a section $\tau(\sigma_{i,j})$ to the π-relative component of S_A it is mapped to under the partial normalization map. This defines an A-structure f on (Γ, τ).



Note that by construction f is generic and is unique up to isomorphisms of A-structures on (Γ, τ). We therefore have a well defined object of $\mathcal{X}$. This defines the functor r on the objects of $\mathcal{F}$.

Let $(\lambda_1\colon C'_A \to C, \lambda_2\colon (S' \to T') \to (S \to T))$ be a morphism over $\lambda\colon B' \to B$ in $\mathcal{F}$ (note that $\xi'_A(\lambda_1) = \phi(\lambda_2)$). By passing through the above construction we see that we get an isomorphism of admissible pairs $(\Gamma', \tau') \to (\Gamma, \tau)$ which commutes with the A-structure f' and f. We get a morphism

$$(S'_{(\Gamma',\tau')} \to T'_{\Gamma'/\tau'}, f') \to (S_{(\Gamma,\tau)} \to T_{\Gamma/\tau}, f).$$

This defines a morphism in $\mathcal{X}$. This completes the definition of the functor $r\colon \mathcal{F} \to \mathcal{X}$.

It remains to check that $r \circ q$ and $q \circ r$ are naturally isomorphic to the respective identity maps. Let $(S_{(\Gamma,\tau)} \to T_{\Gamma/\tau})$ be an object of $\mathcal{X}$, since f is a generic A-structure on (Γ, τ) it is easy to verify that

$$\begin{aligned} r \circ q(S'_{(\Gamma,\tau)} \to T_{\Gamma/\tau}, f) &= r(S_A, S \to T, \mathrm{id}_S) \\ &= (S'_{(\Gamma,\tau)} \to T_{\Gamma/\tau}, f). \end{aligned}$$

If $(S'_A, S \to T, \alpha)$ is an object of $\mathcal{F}$ over B we have

$$\begin{aligned} q \circ r(S'_A, S \to T, \alpha) &= q(S_{(\Gamma,\tau)} \to T_{\Gamma/\tau}, f) \\ &= (S_A, S \to T, \mathrm{id}_S). \end{aligned}$$

The isomorphism α induces an isomorphism $\alpha^{-1}\colon S_A \to S'_A$ so $(S'_A, S \to T, \alpha)$ and $(S_A, S \to T, \mathrm{id}_S)$ are isomorphic by $(\alpha^{-1}, \mathrm{id}_{S\to T})$. We thus have a natural isomorphism $q \circ r \overset{\sim}{\Rightarrow} \mathrm{id}$. $\square$

The Excess Bundle

Let A be a stable $2g+2-4h+2m$ pointed genus g graph. We have shown in Proposition 2.2.18 that the diagram

$$\begin{array}{ccc} \coprod_{(\Gamma,\tau)\in\mathfrak{H}_A} \mathcal{Adm}_{(\Gamma,\tau)} & \xrightarrow{\eta_{\mathfrak{H}}} & \mathcal{Adm}(g,h)_{2m} \\ \phi_{\mathfrak{H}}\downarrow & & \downarrow\phi \\ \overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,2g+2-4h+2m}. \end{array} \tag{2.1}$$

is Cartesian. We will now identify the top Chern class of the excess bundle

$$E = \phi_{\mathfrak{H}}^* N_{\xi_A}/N_{\eta_{\mathfrak{H}}}.$$

Note that in the fiber product of Diagram 2.1 the irreducible components are the same as the connected components. A vector bundle on a space X is the same thing as a collection of vector bundles on the connected components of X. Let (Γ, τ) be an admissible pair with generic A-structure $f = (\alpha, \beta, \gamma)$. We can thus restrict our attention to the diagram

$$\begin{array}{ccc} \mathcal{Adm}_{(\Gamma,\tau)} & \xrightarrow{\eta_\Gamma} & \mathcal{Adm}(g,h)_{2m} \\ \phi_f\downarrow & & \downarrow\phi \\ \overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,n}, \end{array}$$

and compute

$$E_f = \phi_f^* N_{\xi_A}/N_{\eta_\Gamma}$$

the restriction of the excess bundle to $\mathcal{Adm}_{(\Gamma,\tau)}$.

Proposition 2.2.19. With the above notation we have

$$c_{\mathrm{top}} E_f = \prod_{(h,h')\in N} -\phi_f^*(\psi_h + \psi_{h'}). \tag{2.2}$$

where N is a set of representatives of

$$\{e \in E_\Gamma \mid e \neq \tau(e), e \in \operatorname{Im}\beta \cap \tau(\operatorname{Im}\beta)\}$$

under the equivalence relation induced by τ:

$$e \sim e' \qquad \Leftrightarrow \qquad e' = \tau(e).$$

Proof. Recall that fiber products (of stacks) commute with composition. In other words if we have maps $f_1\colon X_1 \to Z$, $f_2\colon X_2 \to X_1$ and $g\colon Y \to Z$ then there is an isomorphism $X_2 \times_{X_1} (X_1 \times_Z Y) \simeq X_2 \times_Z Y$. Moreover if the morphisms f_1 and f_2 are lci, then in the resulting diagram

$$\begin{tikzcd} X_2 \times_{X_1} (X_1 \times_Z Y) \simeq X_2 \times_Z Y \arrow[r, "h_2"] \arrow[d, "g_2"] & X_1 \times_Z Y \arrow[r, "h_1"] \arrow[d, "g_1"] & Y \arrow[d, "g"] \\ X_2 \arrow[r, "f_2"] & X_1 \arrow[r, "f_1"] & Z \end{tikzcd}$$

we have

$$c_{\mathrm{top}}(g_1^* N_{f_1 \circ f_2}/N_{h_1 \circ h_2}) = c_{\mathrm{top}}(g_2^* N_{f_2}/N_{h_2}) \cdot c_{\mathrm{top}}(h_2^*(g_1^* N_{f_1}/N_{h_1}))$$

by the Excess Intersection Formula 1.2.13.

We will argue by induction upon the edges of A. We can decompose the map ξ_A into a sequence of gluing morphisms ξ_i each gluing a single edge. In this way we obtain a sequence of fibered diagrams

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\phi_{\mathfrak{H}}"] & \dots \arrow[r] & F_{i+1} \arrow[r, "\eta_{i+1}"] \arrow[d, "\phi_{i+1}"] & F_i \arrow[r] \arrow[d, "\phi_i"] & \dots \arrow[r] & \mathcal{A}dm(g,h)_{2m} \arrow[d, "\phi"] \\ \overline{\mathcal{M}}_A \arrow[r] & \dots \arrow[r] & \overline{\mathcal{M}}_{A_{i+1}} \arrow[r, "\xi_{i+1}"] & \overline{\mathcal{M}}_{A_i} \arrow[r] & \dots \arrow[r] & \overline{\mathcal{M}}_{g,2g+2-4h+2m} \end{tikzcd} . \tag{2.3}$$

Abusing notation we then have

$$c_{\mathrm{top}}(\phi_{\mathfrak{H}}^* N_{\xi_A}/N_\eta) = \prod c_{\mathrm{top}}(\phi_i^* N_{\xi_{A_i}}/N_{\eta_i}).$$

It therefore suffices to determine $c_{\mathrm{top}}(\phi_i^* N_{\xi_{A_i}}/N_{\eta_i})$ for each i.

By Proposition 2.2.18 the fiber products F_i of Diagram 2.3 take the form

$$F_i = \coprod_{(\Gamma',\tau') \in \mathfrak{H}_{A_i}} \mathcal{A}dm_{(\Gamma',\tau')}$$

Let $\mathcal{A}dm_{\Gamma',\tau'} \subset F_i$ be one of the connected components. The map

$$\phi_f\colon \mathcal{A}dm_{(\Gamma',\tau')} \to \overline{\mathcal{M}}_{A_i}$$

factors as $\xi_{\Gamma' \to A_i} \circ \phi_{\Gamma'}$ where $\phi_{\Gamma'}\colon \mathcal{A}dm_{\Gamma',\tau'} \hookrightarrow \overline{\mathcal{M}}_{\Gamma'}$ is the source map and $\xi_{\Gamma' \to A_i}\colon \overline{\mathcal{M}}_{\Gamma'} \to \overline{\mathcal{M}}_{A_i}$ is the gluing morphism.

We can then form the fibered diagram

$$\begin{tikzcd}
\coprod_{B\in\mathfrak{G}}\coprod_{(\Gamma'',\tau'')\in\mathfrak{H}_B}\mathcal{A}dm_{(\Gamma'',\tau'')} \arrow[r, "\eta"] \arrow[d, "\phi_{\mathfrak{H}}"] & \mathcal{A}dm_{(\Gamma',\tau)} \arrow[d, "\phi_{\Gamma'}"] \\
\coprod_{B\in\mathfrak{G}_{A,\Gamma}}\overline{\mathcal{M}}_B \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{\Gamma'} \arrow[d, "\xi_{\Gamma'\to A_i}"] \\
\overline{\mathcal{M}}_{A_{i+1}} \arrow[r, "\xi_{i+1}"] & \overline{\mathcal{M}}_{A_i}
\end{tikzcd} . \tag{2.4}$$

where the lower square is Cartesian by Proposition 2.1.14 and the upper square is Cartesian by 2.2.18.

Let $\eta''\colon \mathcal{A}dm_{(\Gamma'',\tau'')} \to \mathcal{A}dm_{(\Gamma',\tau')}$ be the restriction of the map η of Diagram 2.4 to a single irreducible component. Since $\operatorname{codim}_{\overline{\mathcal{M}}_{A_i}} \operatorname{Im}(\xi_{i+1}) = 1$ the codimension of $\eta(\mathcal{A}dm_{(\Gamma'',\tau'')})$ in $\mathcal{A}dm_{(\Gamma',\tau')}$ is either 1 or 0. If $\operatorname{codim}_{\mathcal{A}dm_{(\Gamma',\tau')}} \eta(\mathcal{A}dm_{(\Gamma'',\tau'')}) = 1$ then there is no excess on this irreducible component.

If $\operatorname{codim}_{\mathcal{A}dm_{(\Gamma',\tau')}} \eta(\mathcal{A}dm_{(\Gamma'',\tau'')}) = 0$ then $\mathcal{A}dm_{(\Gamma',\tau')} = \mathcal{A}dm_{(\Gamma'',\tau'')}$. Restricting Diagram 2.4 to $\mathcal{A}dm_{(\Gamma',\tau')}$ we get

$$\begin{tikzcd}
\mathcal{A}dm_{(\Gamma'',\tau'')} \arrow[r, "\eta''"] \arrow[d, "\phi_{\Gamma''}"] & \mathcal{A}dm_{(\Gamma',\tau')} \arrow[d, "\phi_{\Gamma'}"] \\
\overline{\mathcal{M}}_{\Gamma'} \arrow[r, "\mathrm{id}"] \arrow[d, "\xi_{\Gamma'\to A_{i+1}}"] & \overline{\mathcal{M}}_{\Gamma'} \arrow[d] \\
\overline{\mathcal{M}}_{A_{i+1}} \arrow[r, "\xi_{i+1}"] & \overline{\mathcal{M}}_{A_i}
\end{tikzcd} .$$

We deduce that

$$\phi^*_{\Gamma''}\xi^*_{\Gamma'\to A_{i+1}}N_{\xi_{i+1}}/N_{\eta''} = \phi^*_{\Gamma''}\left(\xi^*_{\Gamma'\to A_{A_i+1}}N_{\xi_{i+1}}/N_{\mathrm{id}}\right) = \phi^*_{\Gamma''}(\mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee)$$

where (h, h') is the edge of Γ corresponding to the edge of A_{i+1} glued together by the morphism ξ_{i+1} and where the second equality is due to 2.1.18.

Now note that this situation only occurs when A_{i+1} is a specialization of Γ'. In other words this situation occurs exactly if and only if the node glued together by ξ_{i+1} is already a node of Γ, which can only happen if and only if the involution τ on Γ permutes an edge of $\operatorname{Im}\beta$ with another edge of $\operatorname{Im}\beta$, where $\beta\colon H_A \to H_\Gamma$ is the map of half edges induced by $f\colon (\Gamma,\tau) \to A$.

□

2.2.20. Let A be a $n+2m$ pointed genus g stable graph. Let $\phi_n\colon \mathcal{A}dm(g,h)_{2m} \to \overline{\mathcal{M}}_{g,n+2m}$ be the composition of the source map $\phi\colon \mathcal{A}dm(g,h) \to \overline{\mathcal{M}}_{g,2g+2-4h}$ of Definition 1.1.16 and the forgetful map $\pi^{(k)} \colon \overline{\mathcal{M}}_{g,2g+2-4h+2m} \to \overline{\mathcal{M}}_{g,n+2m}$. We form the diagram

$$\begin{tikzcd}
\prod_{B\in\mathfrak{A}}\prod_{(\Gamma,\tau)\in\mathfrak{H}_B} \mathcal{A}dm_{(\Gamma,\tau)} \arrow[r, "\eta_{\mathfrak{H}}"] \arrow[d, "\phi_{\mathfrak{H}}"'] & \mathcal{A}dm(g,h)_{2m} \arrow[d, "\phi"] \\
\prod_{B\in\mathfrak{A}} \overline{\mathcal{M}}_B \arrow[r] \arrow[d, "\pi_{\mathfrak{A}}"'] & \overline{\mathcal{M}}_{g,2g+2-4h+2m} \arrow[d, "\pi^{(k)}"] \\
\overline{\mathcal{M}}_A \arrow[r, "\xi_A"] & \overline{\mathcal{M}}_{g,n+2m}
\end{tikzcd}.$$

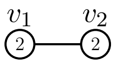
We are now ready to describe the pullback of $\phi_{n*}[\mathcal{A}dm(g,h)_{2m}]$ along the gluing map ξ_A. This is one of the main results of this thesis.

Theorem 2.2.21. With the notation of 2.2.20 and Proposition 2.2.19 we have

$$\begin{aligned}
&\xi_A^*\phi_{n*}([\mathcal{A}dm(g,h)_{2m}]) \\
&= \sum_{B\in\mathfrak{A}_A} \pi_{\mathfrak{A}*}\left(\sum_{\substack{f\colon (\Gamma,\tau)\to B \\ \in\mathfrak{H}_B}} c_{\mathrm{top}}E_f \cap \phi_{\mathfrak{H}*}[\mathcal{A}dm_{(\Gamma,\tau)}]\right).
\end{aligned}$$

Proof. This now follows directly from Corollary 1.2.14, Proposition 2.2.18 and Proposition 2.2.19. ☐

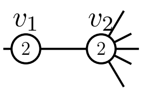
Example 2.2.22. Let A be the stable graph

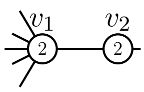


we will use Theorem 2.2.21 to compute $\xi_A^*\phi_{0*}[\mathcal{A}dm(4,1)]$. We form the diagram

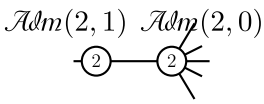
$$\begin{tikzcd}
\coprod_{(\Gamma,\tau,f)\in\mathfrak{H}_B} \mathcal{A}dm_{(\Gamma,\tau)} \arrow[r] \arrow[d] & \mathcal{A}dm(4,1) \arrow[d] \\
\coprod_{B\in\mathfrak{A}} \overline{\mathcal{M}}_B \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{4,6} \arrow[d] \\
\overline{\mathcal{M}}_A \arrow[r] & \overline{\mathcal{M}}_4
\end{tikzcd}$$

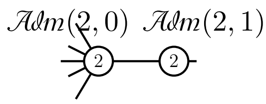
where both squares are Cartesian. The set $\mathfrak{A}$ consists of all possible distributions of legs over the vertices of A. The graphs which will admit an admissible bielliptic pair will have one leg attached to one of the vertices and all other legs to the other vertex.

$B_1 =$ 

$B_2 =$ 

The admissible bielliptic structures that these graphs admit are of the form

$(\Gamma_1,\tau_1) =$ 

$(\Gamma_2,\tau_2) =$ 

and there is only one obvious B_i-structure on (Γ_i, τ_i). Note that the top Chern class of the excess bundle is trivial. We therefore have

$$\xi^* \phi_{0*}[\mathcal{A}dm(4,1)] = \binom{6}{1} \left(\phi_{1*}[\mathcal{A}dm(2,1)] \otimes \phi_{1*}[\mathcal{A}dm(2,0)] + \phi_{1*}[\mathcal{A}dm(2,0)] \otimes \phi_{1*}[\mathcal{A}dm(2,1)] \right).$$

A bielliptic curve of genus 4 has 6 ramification points and every smooth bielliptic curve of genus 4 admits has a unique bielliptic involution (in 3.4.20 we will give a precise argument for this), therefore the degree of $\phi_{0*}\colon \mathcal{A}dm(4,1) \to \overline{\mathcal{M}}_4$ is 6!. Similarly the degree of $\phi_{1*}\colon \mathcal{A}dm(2,1) \to \overline{\mathcal{M}}_{2,1}$ is 1 and the degree of $\phi_{1*}\colon \mathcal{A}dm(2,0) \to \overline{\mathcal{M}}_{2,1}$ is 5!. Thus

$$\begin{aligned}
\xi^*[\overline{\mathcal{B}}_4] &= \frac{1}{6!} \xi^* \phi_{0*}[\mathcal{A}dm(4,1)] \\
&= \frac{\binom{6}{1}}{6!} \left(\phi_{1*}[\mathcal{A}dm(2,1)] \otimes \phi_{1*}[\mathcal{A}dm(2,0)] + \phi_{1*}[\mathcal{A}dm(2,0)] \otimes \phi_{1*}[\mathcal{A}dm(2,1)] \right) \\
&= \frac{1}{5!} \left([\overline{\mathcal{B}}_{2,1}] \otimes 5![\overline{\mathcal{H}}_{2,1}] + 5![\overline{\mathcal{H}}_{2,1}] \otimes [\overline{\mathcal{B}}_{2,1}] \right) \\
&= [\overline{\mathcal{B}}_{2,1}] \otimes [\overline{\mathcal{H}}_{2,1}] + [\overline{\mathcal{H}}_{2,1}] \otimes [\overline{\mathcal{B}}_{2,1}] \in A^3(\overline{\mathcal{M}}_{2,1} \times \overline{\mathcal{M}}_{2,1}).
\end{aligned}$$

This computation will turn out useful later in the computation of $[\overline{\mathcal{B}}_4]$ in Theorem 3.4.8.

Pushing Forward to the Target Space

Let $\phi_n\colon \mathcal{A}dm(g,h)_{2m} \to \overline{\mathcal{M}}_{g,n+2m}$ be the source map, let $\rho\colon \mathcal{A}dm(g,h)_{2m} \to \overline{\mathcal{M}}_{h,2g+2-4h+2m}$ be the target map (see Definition 1.1.16 and Notation 1.3.3) and let A_θ be a decorated boundary graph (see Definition 2.1.21). In this subsection we will identify the pull-push $\rho_* \phi_n^*([A_\theta])$ as a sum of decorated stratum classes in $R^\bullet(\overline{\mathcal{M}}_{h,2g+2-4h+m})$.

Consider the commutative diagram

$$\begin{tikzcd}
\prod_{B\in\mathfrak{A}} \prod_{(\Gamma,\tau)\in\mathfrak{H}_B} \overline{\mathcal{M}}_{\Gamma/\tau} \arrow[r, "\xi_{\Gamma/\tau}"] & \overline{\mathcal{M}}_{h,2g+2-4h+m} \\
\prod_{B\in\mathfrak{A}} \prod_{(\Gamma,\tau)\in\mathfrak{H}_B} \mathcal{A}dm_{(\Gamma,\tau)} \arrow[r, "\eta_{\mathfrak{H}}"] \arrow[u, "\rho_{\mathfrak{H}}"] \arrow[d, "\phi_{\mathfrak{H}}"'] & \mathcal{A}dm(g,h)_{2m} \arrow[u, "\rho"'] \arrow[d, "\phi"] \\
\prod_{B\in\mathfrak{A}} \overline{\mathcal{M}}_B \arrow[r] \arrow[d, "\pi_{\mathfrak{A}}"'] & \overline{\mathcal{M}}_{g,2g+2-4h+2m} \arrow[d, "\pi^{(k)}"] \\
\overline{\mathcal{M}}_A \arrow[r, "\xi_A"] & \overline{\mathcal{M}}_{g,n+2m}
\end{tikzcd} \quad . \tag{2.5}$$

From the excess intersection formula (Proposition 1.2.13) and the proof of Theorem 2.2.21 we have

$$\rho_* \phi^* \pi^{(k)*} \xi_{A*}(\theta \cap [\overline{\mathcal{M}}_A]) = \sum_{(\Gamma,\tau)\in\mathfrak{H}} \xi_{\Gamma/\tau*} \rho_{\mathfrak{H}*} \big(\phi_{\mathfrak{H}}^* \pi_{\mathfrak{A}}^*(\theta) \cdot c_{\mathrm{top}}(E) \cap [\mathcal{A}dm_{(\Gamma,\tau)}] \big).$$

We want to make the right hand side explicit as a sum of decorated stratum classes of $\overline{\mathcal{M}}_{h,2g+2-4h+m}$.

For any decoration θ and a forgetful map $\pi_B\colon \overline{\mathcal{M}}_B \to \overline{\mathcal{M}}_A$ the pullback $\pi_B^*(\theta)$ is given by Lemma 2.2.2 and 2.2.3. For each admissible pair $(\Gamma,\tau) \in \mathfrak{H}_B$ we can restrict $\phi_{\mathfrak{H}}$ and $\rho_{\mathfrak{H}}$ to the maps

$$\overline{\mathcal{M}}_{\Gamma/\tau} \xleftarrow{\rho_\Gamma} \mathcal{A}dm_{(\Gamma,\tau)} \xrightarrow{\phi_f} \overline{\mathcal{M}}_B \ . \tag{2.6}$$

For any decoration $\theta \in A^\bullet(\overline{\mathcal{M}}_B)$ we will find a decoration $\hat{\theta} \in A^\bullet(\overline{\mathcal{M}}_{\Gamma/\tau})$ such that $\phi_f^*(\theta) = \rho_\Gamma(\hat{\theta})$. Since the pullback is a ring homomorphism we can do this for the individual terms of θ. In Proposition 2.2.23 we will show that $\phi_f^*(\psi_h) = \frac{1}{2}\rho_\Gamma^*(\hat{\psi}_i)$, where the half edge or leg h of B corresponds to the half edge or leg i of Γ/τ under the quotient by τ. In Proposition 2.2.24 we will then prove a similar result for the κ classes.

Proposition 2.2.23. Consider the situation of Diagram 2.6 where we denote the A-structure f by $(\alpha,\beta,\gamma)\colon(\Gamma,\tau) \to A$. Let $h \in H_A \cup L_A$ and $i \in H_{\Gamma/\tau} \cup L_{\Gamma/\tau}$ be half edges or legs in the respective graphs such that $\beta(h)$ maps to i under the quotient of τ, i.e. $\operatorname{Orb}_\tau \beta(h) = i$. Let $\psi_h \in A^\bullet(\overline{\mathcal{M}}_A)$ and $\hat{\psi}_i \in A^\bullet(\overline{\mathcal{M}}_{\Gamma/\tau})$ be the respective ψ classes. Then

$$\begin{aligned} 2\phi_f^*(\psi_h) &= \rho_\Gamma^*(\hat{\psi}_i) \qquad \text{if } \tau(\beta(h)) = h \\ \phi_f^*(\psi_h) &= \rho_\Gamma^*(\hat{\psi}_i) \qquad \text{if } \tau(\beta(h)) \neq h. \end{aligned}$$

Proof. Recall from Lemma 2.1.28 that if $f = (\alpha,\beta,\gamma)\colon \Gamma \to A$ is an A-structure on Γ then $\xi_{f\colon \Gamma\to A}(\psi_h) = \psi_{\beta(h)}$. Since $\psi_{\beta(h)}$ is only nontrivial on one component of the product $\overline{\mathcal{M}}_\Gamma = \prod_{v\in V_\Gamma} \overline{\mathcal{M}}_{g(v),n(v)}$ we can restrict to the case where there is just one vertex. In other words consider

$$\overline{\mathcal{M}}_{g,2g+2-4h+2m} \xleftarrow{\phi} \mathcal{A}dm(g,h)_{2m} \xrightarrow{\rho} \overline{\mathcal{M}}_{h,2g+2-4h+m}$$

Let

$$\Xi := \begin{tikzcd} S \arrow[rr, "\mu"] \arrow[dr, "\varrho"] & & T \arrow[dl, "\varpi"'] \\ & B \arrow[ul, bend left, "\sigma_h"] \arrow[ur, bend right, "\hat{\sigma}_i"'] & \end{tikzcd}$$

be an object of $\mathcal{A}dm(g,h)_{2m}$ over B. By definition we have

$$\rho(\Xi) = \begin{tikzcd} T \arrow[d, "\varpi"'] \\ B \arrow[u, bend right, "\hat{\sigma}_i"'] \end{tikzcd}, \quad \phi(\Xi) = \begin{tikzcd} S \arrow[d, "\varrho"'] \\ B \arrow[u, bend right, "\sigma_h"'] \end{tikzcd}.$$

We want to compare the line bundles $\phi^*\mathbb{L}_h$ and $\rho^*\hat{\mathbb{L}}_i$. On the family Ξ we have (see Definition 1.1.34)

$$\begin{aligned} \rho^*(\mathbb{L}_i)_\Xi &= (\mathbb{L}_i)_{\rho(\Xi)} = \sigma_i^*\omega_\varpi \\ \phi_\Gamma^*(\tilde{\mathbb{L}}_h)_\Xi &= (\tilde{\mathbb{L}}_h)_{\phi_\Gamma(\Xi)} = \tilde{\sigma}_h^*\omega_\varrho. \end{aligned}$$

Let R be the ramification divisor of the cover $S \to T$ then on Ξ

$$\begin{aligned}\sigma_i^*(\omega_\varpi) &= \tilde{\sigma}_h^*\mu^*(\omega_\varpi)\\ &= \tilde{\sigma}_h^*(\omega_\varrho(-R))\\ &= \tilde{\sigma}_h^*(\omega_\varrho)\otimes\tilde{\sigma}_h^*(-R)\\ &= \begin{cases}\tilde{\sigma}_h^*(\omega_\varrho)^{\otimes 2} & \text{if } i=\tau(i)\\ \tilde{\sigma}_h^*(\omega_\varrho) & \text{if } i\neq\tau(i)\end{cases}\end{aligned}$$

It follows that

$$\begin{aligned}\phi^*\tilde{\psi}_h &= \rho^*\psi_i &&\text{when } \tau_\beta h\neq h\\ 2\phi^*\tilde{\psi}_h &= \rho^*\psi_i &&\text{when } \tau_\beta(h)=h.\end{aligned}$$

□

Proposition 2.2.24. Consider the situation of Diagram 2.6. Denote by $\kappa_{v,i}$ and $\hat{\kappa}_{v,i}$ the κ classes on $\overline{\mathcal{M}}_A$ and $\overline{\mathcal{M}}_{\Gamma/\tau}$ respectively. For any vertex $v\in\Gamma$ let $N_v = a^{-1}(v)\cup\zeta^{-1}(v)$. We have

$$\phi_f^*(\kappa_{v,i}) = \sum_{w\in\alpha^{-1}(v)}\rho_\Gamma^*\left(2\hat{\kappa}_{w,i} - 2^{-i}\sum_{j\in N_w}\hat{\psi}_{w,j}^i\right).$$

Proof. The proof is a slight generalization of [FP00, pg. 234]. Let $\phi_\Gamma\colon \mathcal{A}dm_{(\Gamma,\tau)}\to\overline{\mathcal{M}}_\Gamma$ be the map sending an admissible cover to the source curve. The map $\phi_f\colon \mathcal{A}dm_{(\Gamma,\tau)}\to\overline{\mathcal{M}}_A$ decomposes as $\phi_\Gamma\circ\xi_{\Gamma\to A}$. We already know that

$$\xi_{\Gamma\to A}^*(\kappa_{v,i}) = \sum_{w\in\alpha^{-1}(v)}\kappa_{w,i}$$

by Lemma 2.1.28.

It remains to identify $\phi_\Gamma^*(\kappa_{v,i})$. There are two cases, either the vertex v of Γ is fixed under the involution τ or it is not.

Suppose v is fixed under τ. Consider the diagram

$$\begin{tikzcd}\overline{\mathcal{M}}_{g,2g+2-4h+2m+1} \arrow[d, "r"] & \mathcal{S} \arrow[l, "\iota"'] \arrow[r, "f"] \arrow[d, "q"] & \overline{\mathcal{M}}_{h,2g+2-4h+m+1} \arrow[d, "p"]\\ \overline{\mathcal{M}}_{g,2g+2-4h+2m} & \mathcal{A}dm(g,h)_{2m} \arrow[l, "\phi"] \arrow[r, "\rho"'] & \overline{\mathcal{M}}_{h,2g+2-4h+m}\end{tikzcd}$$

where $\overline{\mathcal{M}}_{g,2g+2-4h+2m+1}$ is viewed as the universal curve of $\overline{\mathcal{M}}_{g,2g+2-4h+2m}$, where $\mathcal{S}$ is the restriction of $\overline{\mathcal{M}}_{g,2g+2-4h+2m+1}$ to $\mathcal{A}dm(g,h)_{2m}$ under the inclusion ϕ, $\overline{\mathcal{M}}_{h,2g+2-4h+m+1}$ is viewed as the universal curve of $\overline{\mathcal{M}}_{h,2g+2-4h+m}$ and f is the induced 2-to-1 map.

Let $\psi_\bullet$ and $\hat{\psi}_\bullet$ be the ψ classes of the last point of $\overline{\mathcal{M}}_{g,2g+2-4h+1}$ and $\overline{\mathcal{M}}_{h,2g+2-4h+m+1}$ respectively. Let D be the branch divisor of f. By Riemann-Hurwitz we have $\iota^*(\psi_\bullet) =$

$f^*(\hat{\psi}_\bullet - D/2)$. We now have

$$\begin{aligned}
\phi^*(\kappa_i) &= \phi^* r_*(\psi_\bullet^{i+1}) \\
&= q_* \iota^*(\psi_\bullet^{i+1}) \\
&= q_* f^*((\hat{\psi}_\bullet - D/2)^{i+1}) \\
&= 2\rho^* p_*((\hat{\psi}_\bullet - D/2)^{i+1}) \\
&= 2\rho^* p_*(\hat{\psi}_\bullet^{i+1} + (-D/2)^{i+1}) \\
&= \rho^* \left(2\hat{\kappa}_i + 2 \sum_{j=1}^{2g+2} (-\frac{1}{2})^{i+1} (-\hat{\psi}_j)^i \right) \\
&= \rho^* \left(2\hat{\kappa}_i - 2^{-i} \sum_{j=1}^{2g+2} \hat{\psi}_j^i \right).
\end{aligned}$$

Now suppose that v is not fixed by τ and consider now the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{h,n+1} \times \overline{\mathcal{M}}_{h,n} \amalg \overline{\mathcal{M}}_{h,n} \times \overline{\mathcal{M}}_{h,n+1} \arrow[d, "r"] & \mathcal{S} \arrow[l, "\iota"'] \arrow[r, "f"] \arrow[d, "q"] & \overline{\mathcal{M}}_{h,n+1} \arrow[d, "p"] \\
\overline{\mathcal{M}}_{h,n} \times \overline{\mathcal{M}}_{h,n} & \overline{\mathcal{M}}_{h,n}^D \arrow[l, "\phi"] \arrow[r, "\rho"'] & \overline{\mathcal{M}}_{h,2g+2}
\end{tikzcd}$$

In this case we have $\phi^* \circ p_1^*(\tilde{\kappa}_i) = \rho^*(\kappa_i)$ where $p_1 \colon \overline{\mathcal{M}}_{h,n} \times \overline{\mathcal{M}}_{h,n} \to \overline{\mathcal{M}}_{h,n}$ is the first projection map.

Now for a vertex $w \in \Gamma$ with $\tau(w) = w$ let $\phi_w \colon \overline{\mathcal{H}}_{g(w),n(w)-2g-2}$. We have

$$\begin{aligned}
\phi_\Gamma^*(\tilde{\kappa}_{v,i}) &= \sum_{w \in \alpha^{-1}(v)} \phi_w^*(\tilde{\kappa}_{w,i}) \\
&= \sum_{\substack{w \in \alpha^{-1}(v) \\ \tau(w)=w}} \left(2\rho_\Gamma^*(\kappa_{v_\tau,i}) - 2^{-i} \sum_{\substack{h \in a_\Gamma^{-1}(v) \\ \tau(h)=h}} \rho_\Gamma^*(\psi_{w,i}) \right) + \sum_{\substack{w \in \alpha^{-1}(v) \\ \tau(w) \neq w}} \rho_\Gamma^*(\kappa_{v_\tau,i}).
\end{aligned}$$

□

Notation 2.2.25. Let (Γ, τ) be an admissible pair with an A-structure f. We will denote by F the function that assigns to any decoration θ on A the decoration $F(\theta)$ on Γ/τ such that

$$\phi_f^*(\theta) = \rho_\Gamma^*(F(\theta)).$$

Proposition 2.2.23 and 2.2.24 ensure that F is well defined (and give an explicit combinatorial description of F).

Theorem 2.2.26. Let A_θ be an $n+2m$ pointed genus g decorated boundary graph. Let $\mathcal{A}dm(g,h)_{2m}$ be a space of admissible covers and ϕ_n and ρ be the source and target maps (as in Definition 1.1.16 and Notation 1.3.3). Let F be as in 2.2.25. In $A^\bullet(\overline{\mathcal{M}}_{h,2g+2-4h+m})$ we have

$$\begin{aligned}
&\rho_* \phi^*([A_\theta]) \\
&= \frac{1}{|\operatorname{Aut} A|} \sum_{B \in \mathfrak{A}_A} \sum_{\substack{f \colon (\Gamma,\tau) \to B \\ \in \mathfrak{H}_B}} \deg \rho_\Gamma \, \xi_{\Gamma/\tau *} \Big(F(\theta \cdot c_{\mathrm{top}}(E_f)) \cap [\overline{\mathcal{M}}_{\Gamma/\tau}] \Big)
\end{aligned}$$

where E_f is as in Proposition 2.2.19.

Proof. We push forward the expression from Theorem 2.2.21 through ρ. $\square$

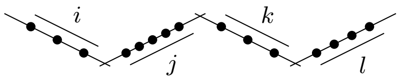
Corollary 2.2.27. If $\mathcal{Adm}(g,h)_{2m}$ is a space of admissible covers and ϕ_n and ρ from Definition 1.1.16 and Notation 1.3.3 are the source and target maps from $\mathcal{Adm}(g,h)_{2m}$ then in particular Theorem 2.2.26 implies that the pull-push

$$\rho_*\phi_n^*\colon A^\bullet(\overline{\mathcal{M}}_{g,n+2m}) \to A^\bullet(\overline{\mathcal{M}}_{h,2g+2-4h+m})$$

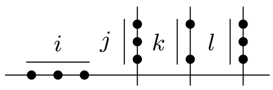
sends tautological classes to tautological classes.

Notation 2.2.28. In all the cases where we will use Theorem 2.2.26 later in this thesis we will have $h = 0$. The image of $\rho_*\phi_n^*$ lies in the $\mathfrak{S}_{2g+2-n}$ invariant part of $A^\bullet(\overline{\mathcal{M}}_{0,2g+2})$. We will therefore introduce some notation for elements in this ring.

Denote by $d_{i_0,\ldots,i_k} \in A^k(\overline{\mathcal{M}}_{0,n})^{\mathfrak{S}_n}$ the class of the closure of all curves in $\overline{\mathcal{M}}_{0,n}$ consisting of a string of k genus 0 curves with i_0 marked points on the first component etc. We define $e_{i,j,k,l} \in A^3(\overline{\mathcal{M}}_{0,n})^{\mathfrak{S}_n}$ as the class of the closure of all curves with one genus 0 component C_0 with three genus zero components C_1, C_2, C_3 attached to it and with i points on C_0, j points on C_1 etc. In other words the curves are of the following form:



curves in $d_{i,j,k,l}$



curves in $e_{i,j,k,l}$.

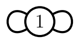
We adopt similar notation for $A^k(\overline{\mathcal{M}}_{0,n})^{\mathfrak{S}_{n-1}}$ where the component with the marked point not acted upon by the symmetric group is marked with $\bullet$, i.e. $d_{2,2\bullet,2}$ indicates the closure of the locus of all stable curves of genus 0 with 7 marked points having 3 components C_0, C_1, C_3 with the point marked by $\bullet$ on the middle component and each component containing 2 of the other marked points.

The invariant cycles of $A^k(\overline{\mathcal{M}}_{0,n})$ we just introduced additively generate $A^k(\overline{\mathcal{M}}_{0,n})$ when k is at most 3. When the codimension is higher this is no longer the case.

2.3 Examples

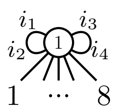
In this section we will give a some concrete examples of Theorem 2.2.21 and Theorem 2.2.26.

Example 2.3.1. Let

$A =$ 

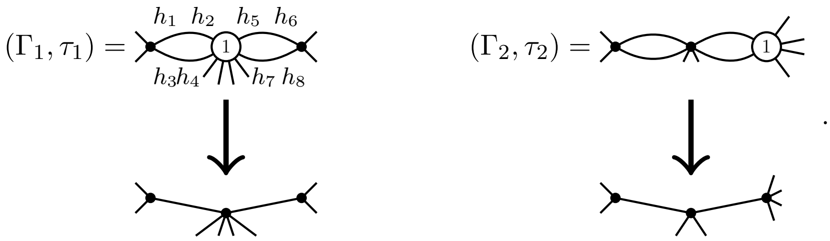
and consider the maps $\phi_0\colon \mathcal{Adm}(3,0) \to \overline{\mathcal{M}}_3$, $\rho\colon \mathcal{Adm}(3,0) \to \overline{\mathcal{M}}_{0,8}$. We compute $\rho_*\phi_0^*([A])$ using Theorem 2.2.26. We will go step by step through the different parts of the equation

- We have $\#\operatorname{Aut} A = 8$.
- The set $\mathfrak{A}$, introduced in 2.2.1, consists of all possible distributions of the legs $\{1, \ldots, 8\}$ over the vertices of A. Since A has only one vertex, the set $\mathfrak{A}_A$ consist of a single element.

$\mathfrak{A} = \Bigg\{ B :=$ 

$\Bigg\}$

- There are two possible types of admissible hyperelliptic pairs with a generic B-structure.



  There are $\frac{1}{2}\binom{8}{2,4,2}$ possible nonisomorphic distributions of the legs for the graph Γ_1 and $\binom{8}{2,4,2}$ for Γ_2.

- There are 8 isomorphism classes of B-structures on (Γ_1, τ). For any generic A-structure f' on (Γ_1, τ) there exists an automorphism of A-structures $(\Gamma_1, \tau, f') \to (\Gamma_1, \tau, f)$ with $f = (\alpha, \beta, \gamma)$ such that the edges of B are mapped to the edges (h_1, h_2) and (h_5, h_6) under β. There are 4 possible choices for the image of the half edge i_1 and given this choices there are 2 possible images for the halfedge i_3. This completely determines a B-structure on (Γ, τ). A similar argument shows that there are 8 isomorphism classes of B-structures for Γ_2.

- We have $\deg \rho_{(\Gamma_1,\tau_1)} = \deg \rho_{(\Gamma_2,\tau_2)} = 2$ by Proposition 1.2.3.

- For all generic B-structures f on Γ_1 and Γ_2, the set $\operatorname{Im} \beta_f \cap \tau(\operatorname{Im} \beta_f)$ is empty, so the top Chern class of the excess bundle is 1.

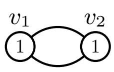
- The graph A is undecorated, in other words $\theta = 1$. Therefore we have $\rho_{(\Gamma_i,\tau_i)*}\phi^*_{f_i}\pi^*_B(\theta) = 1$.

- The image of $\rho_*\phi_0^*\colon A^\bullet(\overline{\mathcal{M}}_3) \to A^\bullet(\overline{\mathcal{M}}_{0,8})$ lies in the $\mathfrak{S}_8$ invariant part $A^\bullet(\overline{\mathcal{M}}_{0,8})^{\mathfrak{S}_8}$ of $A^\bullet(\overline{\mathcal{M}}_{0,8})$. Denote by $d_{2,4,2}, d_{4,2,2} \in A^\bullet(\overline{\mathcal{M}}_{0,8})^{\mathfrak{S}_8}$ the classes given by taking the sum over all nonisomorphic dual graphs of the form (respectively):

  Note that there are $\frac{1}{2}\binom{8}{2,4,2}$ irreducible boundary components in $d_{2,4,2}$ and $\binom{8}{2,4,2}$ in $d_{4,2,2}$. Putting everything together we have

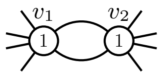
$$\begin{aligned}
\rho_*\phi_0^*([A]) &= \frac{1}{8}\cdot 1\left(\frac{1}{2}\binom{8}{2,4,2}\cdot 16\cdot 2\binom{8}{2,4,2}^{-1} d_{2,4,2} + \binom{8}{2,4,2}\cdot 16\cdot\binom{8}{2,4,2}^{-1} d_{2,4,2}\right)\\
&= 2d_{2,4,2} + 2d_{4,2,2}
\end{aligned}$$

Example 2.3.2. Let

$A :=$ 

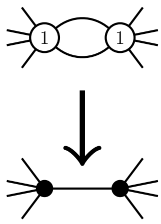
- We have $\#\operatorname{Aut} A = 4$.

- The set $\mathfrak{A}_A$ consists of all possible distributions of points among the vertices of A. The graphs in $\mathfrak{A}_A$ for which there is an admissible hyperelliptic pair are of the form



There are $\binom{8}{4}$ such graphs.

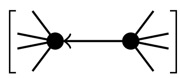
- There is only one admissible hyperelliptic pair (Γ, τ) that admits a B-structure:

$(\Gamma, \tau) =$ 

There is only one isomorphism class of B-structures on (Γ, τ).

- We have $\deg \rho_{(\Gamma,\tau)} = 1$.
- For each f the set $\operatorname{Im} \beta_f \cap \tau(\operatorname{Im} \beta_f)$ consists of all the halfedges of Γ.
- In conclusion we get

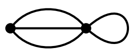
$$\rho_* \phi_0^*([A]) = -\frac{1}{4}\binom{8}{4} 2$$



$$= -\frac{1}{4} d_{4,1,3} - \frac{1}{12} d_{4,2,2}$$

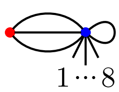
where we used Proposition 1.3.9 to compute the ψ class for the second equality.

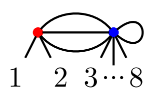
Example 2.3.3. Let

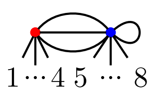
$A :=$  .

We will calculate its image in $A_4(\overline{\mathcal{M}}_{0,8})$.

- We have $\# \operatorname{Aut} A = 3! \cdot 2 = 12$.
- The set $\mathfrak{A}_A$ consists of all distributions of 8 points on A. The elements $B \in \mathfrak{A}_A$ such that there is an admissible hyperelliptic pair with a B-structure are of the form

$B_1 =$ 

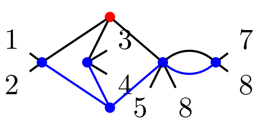
$B_2 =$ 

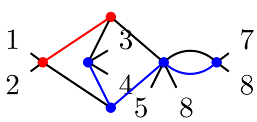
$B_3 =$ 

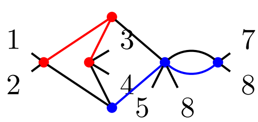
.

There are in total 1 graph of type B_1, $\binom{8}{2}$ graphs of type B_2 and $\binom{8}{4}$ graphs of type B_3.

- The corresponding hyperelliptic pairs are given by

$\Gamma_1 =$ 

$\Gamma_2 =$ 

$\Gamma_3 =$ 

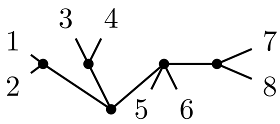
Where τ is the graph involution exchanging the upper half with the bottom half. One way to form an B_i-structure $f = (\alpha, \beta, \gamma)$ on (Γ_i, τ) is by sending all the blue (resp. red) vertices of Γ_i to blue (resp. red) vertices of B_i under α and sending the blue (resp. red) edges of Γ_i to blue (resp. red) vertices to B_i under γ. There are respectively

$$12 \cdot \tfrac{1}{2}\tbinom{8}{2,2,2,2}, \quad 12 \cdot \tbinom{6}{2,2,2}, \quad 12 \cdot \tfrac{1}{2}\tbinom{4}{2}^2$$

isomorphism classes of B_i-structures to (Γ_i, τ).

- Again we immediately verify that $\mathrm{Im}(\beta_f) \cap \tau(\mathrm{Im}\,\beta_f) = \emptyset$ for all the B_i-structures f on Γ_i. Therefore the top Chern class of the excess bundle is 1.

- The quotient graph Γ_i/τ is the same in each case and of the form:

$\Gamma_i/\tau =$  .

In each case we have $\deg \rho_{\Gamma_i/\tau} = 2$.

- Let

$$e = \sum_G \xi_{G*}[\overline{\mathcal{M}}_G] \in A^4(\overline{\mathcal{M}}_{0,8})^{\mathfrak{S}_8}$$

where the sum is taken over all different distributions of the legs of the graph Γ_i/τ. Then adding all of the above together we get

$$\rho_* \phi_0^*[A] = 8e.$$

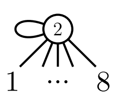
Example 2.3.4. Let

$A_\theta =$ 

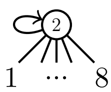
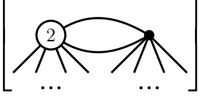
In other words $[A_\theta] = \frac{1}{2}\xi_{A*}(\psi_{h_1})$ where ξ_A is the gluing map $\overline{\mathcal{M}}_{2,2} \to \overline{\mathcal{M}}_3$

- $\#\,\mathrm{Aut}\,A = 2$

- We have

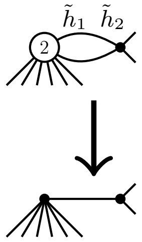
$\mathfrak{A}_A = \Bigg\{$  $\Bigg\}$.

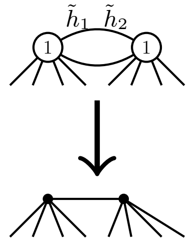
- By the extended comparison result (see 2.2.2) we have $\pi_B^* \psi_{h_1} = \psi_{h_1} - D_{h_1}$ where D_{h_1} is the sum of all divisors with h_1 and any of the points $1, ..., 8$ on an irreducible component of genus 0. In other words

$\pi_B^*(\psi_{h_1}) = \psi_{h_1} - D_{h_1} = \Bigg[$  $\Bigg] - \sum \Bigg[$  $\Bigg]$

Let us call the decorated graph on the left $B_{1,\theta}$ and graphs on the right B_2.

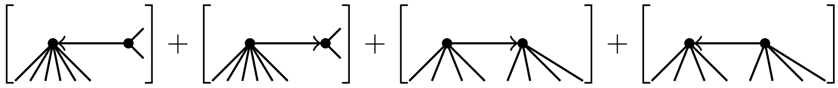
- The graphs which admit a B_1-structure are

$(\Gamma_{1,1}, \tau_{1,1}) =$ 

$(\Gamma_{1,2}, \tau_{1,2}) =$ 

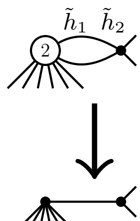
.

In both cases there is one isomorphism class of B_1-structures f such that $f(\tilde{h}_1) = h_1$ and one such that $f(\tilde{h}_2) = h_2$ for each possible distribution of the points. We therefore have

$\rho_*\phi^*[B_{1,\theta}] =$ 

$$
\begin{aligned}
&= \frac{1}{3}d_{5,1,2} + \frac{1}{5}d_{4,2,2} + \frac{1}{10}d_{3,3,2} + \frac{1}{15}d_{2,4,2} \\
&\quad + 0 \\
&\quad + \frac{1}{12}d_{4,2,2} + \frac{1}{4}d_{4,1,3}
\end{aligned}
$$

- The only graphs of type B_2 which admit an hyperelliptic pair are those with two legs on the rational component on the right and all other legs on the genus 2 curve. The only admissible hyperelliptic pair is of the form

$(\Gamma_2, \tau_2) =$ 

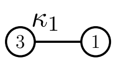
and there is only 1 isomorphism class of B_2-structures. The excess bundle is given by $-\frac{1}{2}(\psi_{\tilde{h}_1} + \psi_{\tilde{h}_2})$. We therefore have

$$\rho_*\phi_*(D_{h_1}) = \frac{1}{3}d_{5,1,2} + \frac{1}{5}d_{4,2,2} + \frac{1}{10}d_{3,3,2} + \frac{1}{15}d_{2,4,2}$$

- Putting everything together we get

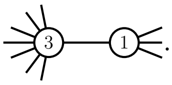
$$\rho_*\phi_n^*[A_\theta] = \frac{2}{3}d_{5,1,2} + \frac{29}{60}d_{4,2,2} + \frac{1}{4}d_{4,1,3} + \frac{1}{5}d_{3,3,2} + \frac{2}{15}d_{2,4,2}$$

Example 2.3.5. Let

$A_\theta =$ 

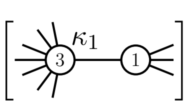
- $\#\,\mathrm{Aut}\,A = 1$.

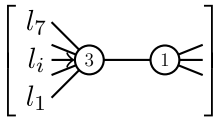
- The only graphs in $\mathfrak{A}_A$ for which there exist admissible hyperelliptic pairs are of the form

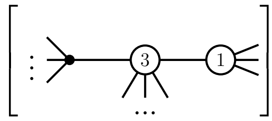
$B =$  .

Indeed any hyperelliptic pair with a B structure must have at least 4 legs + half edges incident to a vertex of genus 1 and at least 8 half edges incident to a vertex of genus 3.

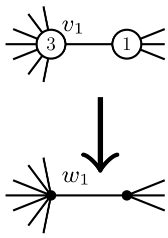
- The pullback of κ_1 to B is

$\pi_B^*(\kappa_1) =$ 

$- \sum_{i=1}^{7}$ 

$+ \sum$ 

- None of the components in the second sum admit a hyperelliptic pair (because the vertex of genus 3 will have less then 8 legs and half edges incident to it). We can therefore restrict to pulling back $[B_\theta]$ where $\theta = \kappa_{v_1} + \sum_{i=1}^{7} \psi_{l_i}$. The only admissible pair with a B-structure is

$(\Gamma, \tau) =$ 

where the action of τ is trivial. It is easy to see there is no excess of intersection.

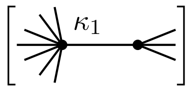
- We have $\deg \rho_{\Gamma/\tau} = 1/4$.

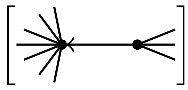
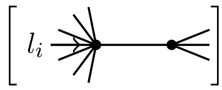
- $\phi_f^*(\kappa_{v_1,1})$ is given by Proposition 2.2.24 we have

$$\phi_f^*(\kappa_{v_1,1}) = 2\rho^*_{(\Gamma,\tau)}(\kappa_{w_1,1}) - \frac{1}{2}\rho^*_{(\Gamma,\tau)}(\psi_h) - \frac{1}{2}\sum_{i=1}^{7} \rho^*_{(\Gamma,\tau)}(\psi_{\tau,l_i})$$

$$\sum_{i=1}^{7} \phi_f^*(\psi_{l_i}) = \frac{1}{2}\sum_{i=1}^{7} \rho_f^*(\psi_{\tau,l_i})$$

In other words we have

$\rho_*\phi^*\pi^*([A]) = \sum \Big(\frac{1}{2}$ 

$- \frac{1}{8}$ 
$- \frac{1}{4}\sum_{i=1}^{7}$ 

$\Big)$

where the outer sum is taken over all distinct distributions of the legs.

–III–

Computing Classes of Spaces of Admissible Double Covers

Assume that the class $[\phi_n(\mathcal{Adm}(g,h)_{2m})]$ of a space of admissible covers in $\overline{\mathcal{M}}_{g,n+2m}$ is tautological. In this chapter we will use Theorems 2.1.30, 2.2.21 and 2.2.26 to compute the class $[\phi_n(\mathcal{Adm}(g,h)_{2m})]$ in terms of decorated stratum classes. We shall discuss four different techniques. We will explicitly compute the classes of $\overline{\mathcal{H}}_5$, $\overline{\mathcal{B}}_{2,1,0}$, $\overline{\mathcal{B}}_{2,0,2}$ and $\overline{\mathcal{B}}_4$ which did not appear earlier in the literature.

3.1 Perfect Pairing

The intersection pairing $H^k(\overline{\mathcal{M}}_{g,n}) \otimes H^{2\dim \overline{\mathcal{M}}_{g,n}-k}(\overline{\mathcal{M}}_{g,n}) \to \mathbb{C}$ is perfect since $\overline{\mathcal{M}}_{g,n}$ is a smooth complete Deligne-Mumford stack. The Gorenstein conjecture asks whether this pairing restricted to the tautological rings $RH^{2k}(\overline{\mathcal{M}}_{g,n}) \otimes RH^{2\dim \overline{\mathcal{M}}_{g,n}-2k}(\overline{\mathcal{M}}_{g,n})$ is perfect as well. As was first shown in [PT14] the Gorenstein conjecture is false in general. However for low g, n and k it is known to hold (for example because it is known that $H^{2k}(\overline{\mathcal{M}}_{g,n}) = RH^{2k}(\overline{\mathcal{M}}_{g,n})$).

Let $\phi_n \colon \mathcal{Adm}(g,h)_{2m} \to \overline{\mathcal{M}}_{g,n+2m}$ be the source map as in Definition 1.1.16. In cases where the intersection pairing of $\overline{\mathcal{M}}_{g,n+2m}$ restricted to the tautological ring is perfect and where the class $[\phi_n(\mathcal{Adm}(g,h)_{2m})]$ of a space of admissible double covers is tautological, we can compute this class in the following way. Let k be the codimension of $\phi_n(\mathcal{Adm}(g,h)_{2m})$ in $\overline{\mathcal{M}}_{g,n+2m}$. Suppose we have fixed bases $\{D_i\}$ and $\{\hat{D}_j\}$ for $RH^{2k}(\overline{\mathcal{M}}_{g,n+2m})$ and for $RH^{2\dim \overline{\mathcal{M}}_{g,n+2m}-2k}(\overline{\mathcal{M}}_{g,n+2m})$ where D_i and $\hat{D}_j$ are decorated stratum classes. Using Theorem 2.1.30 we can compute the intersection matrix

$$M = (D_i \cdot \hat{D}_j)_{ij}.$$

We want to find an expression $[\phi_n(\mathcal{A}dm(g,h)_{2m})] = \sum a_i D_i$. We have

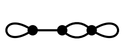
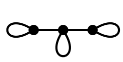
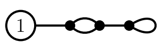
$$([\phi_n(\mathcal{A}dm(g,h)_{2m})] \cdot \hat{D}_j)_j = \left(\sum_i a_i D_i \cdot \hat{D}_j\right)_j .$$

We can compute the numbers on the left hand side using Theorem 2.2.21 or Theorem 2.2.26. Since M is an invertible matrix by assumption, we can solve this expression for the a_i.

Note that when the intersection product is perfect on $R^\bullet(\overline{\mathcal{M}}_{g,n})$ we can run the same argument to obtain the class in the tautological Chow ring. (However, there are no cases when we know that this is the case except when $R^\bullet(\overline{\mathcal{M}}_{g,n}) = RH^{2\bullet}(\overline{\mathcal{M}}_{g,n}) = H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$.)

Example 3.1.1. We can use this method to compute the class of the hyperelliptic locus in the genus 3 case explicitly.

A basis for $A^1(\overline{\mathcal{M}}_3)$ is given by λ, δ_0 and δ_1. A basis in terms of decorated stratum classes for $A^5(\overline{\mathcal{M}}_3)$ is given by (see [Fab90a])

$\hat{D}_1 = \Big[$  $\Big]$ $\hat{D}_2 = \Big[$  $\Big]$ $\hat{D}_3 = \Big[$  $\Big]$

The intersection numbers between the bases have been computed in [Fab90a, pg. 418] (or use Theorem 2.1.30):

$$\begin{array}{ccc} \lambda \cdot \hat{D}_1 = 0 & \lambda \cdot \hat{D}_2 = 0 & \lambda \cdot \hat{D}_3 = \frac{1}{96} \\ \delta_0 \cdot \hat{D}_1 = -\frac{1}{4} & \delta_0 \cdot \hat{D}_2 = 0 & \delta_0 \cdot \hat{D}_3 = \frac{1}{8} \\ \delta_1 \cdot \hat{D}_1 = \frac{1}{8} & \delta_1 \cdot \hat{D}_2 = -\frac{1}{16} & \delta_1 \cdot \hat{D}_3 = -\frac{1}{96} \end{array}$$

Recall that we denote by $[\overline{\mathcal{H}}_3]$ the class $\frac{1}{8!}\phi_{0*}[\mathcal{A}dm(3,0)]$. We leave it as an exercise to the reader to compute the intersection numbers

$$[\overline{\mathcal{H}}_3] \cdot \hat{D}_1 = -\tfrac{1}{8} \qquad [\overline{\mathcal{H}}_3] \cdot \hat{D}_2 = \tfrac{3}{16} \qquad [\overline{\mathcal{H}}_3] \cdot \hat{D}_3 = 0$$

using Theorem 2.2.21 or Theorem 2.2.26. Solving the resulting system of equations we get the well known expression

$$[\overline{\mathcal{H}}_3] = (9\lambda - \delta_0 - 3\delta_1) \in A^1(\overline{\mathcal{M}}_3)$$

Remark 3.1.2. The Sage program by Schmitt has this technique implemented as a function. However it is computationally not the most efficient method. This method can compute $[\overline{\mathcal{B}}_{2,2,0}]$ in a short amount of time but most classes which are more complicated require too much computing time and memory. The problem is that in computing the intersection of $\phi_{n*}[\mathcal{A}dm(g,h)]$ with a decorated stratum classes $[A_\theta]$ we first have to pull back $[A_\theta]$ through the forgetful map $\pi^{(k)}\colon \overline{\mathcal{M}}_{g,2g+2-4h} \to \overline{\mathcal{M}}_{g,n}$ in the decomposition $\phi_n = \pi^{(k)} \circ \phi$. If the number $2g+2-4h$ of ramification points of admissible covers in $\mathcal{A}dm(g,h)$ is very high then the pullback $\pi^{(k)*}[A_\theta]$ in terms of decorated stratum classes involves a large number of different classes. This makes the calculation computationally hard when $2g+2-4h$ is large.

3.2 Pulling Back from Higher Genus

In this section we will use Theorem 2.2.21 to show that if we have an expression for $[\mathcal{A}dm(g,h)_{2m}]$ in terms of decorated stratum classes then by performing pullbacks along boundary maps we can use this to obtain expressions for $[\phi_{n'}(\mathcal{A}dm(g',h)_{2m'})]$ in terms of decorated stratum classes when $g' \leq g$ and $g'+n'+m' \leq g+n+m$.

Let $\phi_n \colon \mathcal{A}dm(g,h)_{2m} \to \overline{\mathcal{M}}_{g,n+m}$ be the composition of the source map of the admissible cover and the forgetful map as in Definition 1.1.16. Suppose that the class $\phi_{n*}[\mathcal{A}dm(g,h)_{2m}]$ is tautological and that we have some expression of it in terms of decorated stratum classes. Suppose we want to know an expression of the class $\phi_{n+1*}([\mathcal{A}dm(g-1,h)_{2m}])$ or $\phi_{n*}([\mathcal{A}dm(g-1,h)_{2m+2}])$ in terms of decorated stratum classes. Consider the graphs

. (3.1)

Let A be either of these graphs. The pullback $\xi_A^*\phi_{n*}(\mathcal{A}dm(g,h)_{2m})$ can be computed using Theorem 2.2.21. In other words we can identify the diagrams

$$\begin{array}{ccc} \coprod_{B\in\mathfrak{A}}\coprod_{(\Gamma,\tau,f)\in\mathfrak{H}_B}\mathcal{A}dm_{(\Gamma,\tau)} & \xrightarrow{\eta_\Gamma} & \mathcal{A}dm(g,h) \\ \downarrow{\scriptstyle \phi_f} & & \downarrow{\scriptstyle \phi} \\ \coprod_{B\in\mathfrak{A}}\overline{\mathcal{M}}_B & \xrightarrow{\pi_B} & \overline{\mathcal{M}}_{g,2g+2-4h+2m} \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi^{(k)}} \\ \overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,n+2m} \end{array} \tag{3.2}$$

and identify the restriction E_Γ of the excess bundle to each component $\mathcal{A}dm_{(\Gamma,\tau)}$. We then have

$$\xi_A^*\phi_{n*}(\mathcal{A}dm(g,h)_{2m}) = \sum_{B\in\mathfrak{A}}\sum_{f\in\mathfrak{H}_B}\pi_{B*}\phi_{f*}(c_{\mathrm{top}}(E_\Gamma)\cap[\mathcal{A}dm_{(\Gamma,\tau)}]) \tag{3.3}$$

If A is the graph on the left of (3.1) then one of the factors is $\pi_{B*}\phi_{f*}[\mathcal{A}dm(g-1,h)_{2m} \times \mathcal{A}dm(1,0)]$. Note that $\phi_{1*}[\mathcal{A}dm(1,0)] = [\overline{\mathcal{M}}_{1,1}]$ so

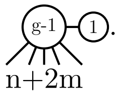
$$\phi_{f*}[\mathcal{A}dm(g-1,h)_{2m} \times \mathcal{A}dm(1,0)] = \phi_{n+1*}[\mathcal{A}dm(g,h)_{2m}] \otimes [\overline{\mathcal{M}}_{1,1}].$$

The hope is that we can compute all the other summands on the right hand side of Equation (3.3) in terms of decorated stratum classes. The pullback map $\xi_A^*\colon R^k(\overline{\mathcal{M}}_{g,n+2m}) \to R^*(\overline{\mathcal{M}}_A)$ can be given, in terms of a basis of decorated stratum classes, by using Theorem 2.1.30. Since we assume the class $[\mathcal{A}dm(g,h)_{2m}]$ is known in terms of decorated stratum classes this gives an expression for $\xi_A^*[\mathcal{A}dm(g,h)_{2m}]$ in terms of decorated stratum classes. From the Equality (3.3) we then obtain an expression for $\phi_{n+1*}[\mathcal{A}dm(g-1,h)_{2m}]$ in terms of decorated stratum classes.

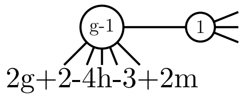
Similarly if A is the graph on the right of (3.1) then one of the factors in the sum of (3.3) is $\phi_{n*}[\mathcal{A}dm(g-1,h)_{2m+2}]$ and if we know all other components in the sum of Equation (3.3) we obtain an expression for this class in terms of decorated stratum classes.

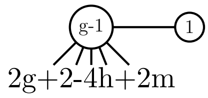
We will now compute the sum of Equation (3.3) in these two cases. As an example, using an expression in terms of decorated stratum classes of $[\overline{\mathcal{B}}_3] := [\phi_0(\mathcal{A}dm(3,1))]$, we will then use this to compute the classes $[\overline{\mathcal{B}}_{2,1,0}] := [\phi_1(\mathcal{A}dm(2,1))]$ and $[\overline{\mathcal{B}}_{2,0,2}] := [\phi_0(\mathcal{A}dm(2,1)_2)]$.

3.2.1. Let

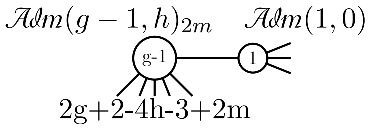
$A =$ 

We will now identify the fiber Diagram 3.2. The set $\mathfrak{A}$ consists by definition of the stable graphs where we distribute the missing $2g+2-4h-n$ legs in all possible ways over the vertices of A_1. There are two types of distributions which admit an admissible pair.

$B_1 =$ 

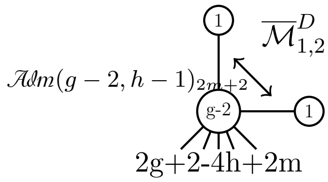
$B_2 =$ 

Graphs of type B_1 admit admissible pairs of the form

$(\Gamma_1, \tau_1) =$ 

Up to isomorphism there is one B_1-structure f_1 on (Γ_1, τ_1).

The only admissible pair which admits a B_2-structure is

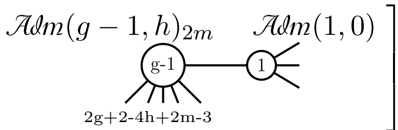
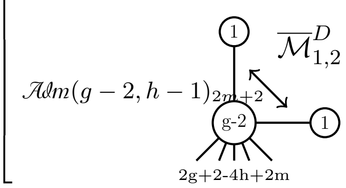
$(\Gamma_2, \tau_2) =$ 

Where $\overline{\mathcal{M}}^D_{1,2}$ is defined as in Notation 1.1.17. Again up to isomorphism there is only one B_2-structure f_2 on (Γ_2, τ_2), it sends the two vertices of Γ_2 on the left to the vertex of genus $g-1$ of B_2 and the vertex of Γ_2 on the right to the vertex of B_2 of genus 1.

In both cases the top Chern class of the excess bundle E_Γ is 1 (see Proposition 2.2.19). Note that $\phi_{1*}[\mathcal{A}\!\mathit{dm}(1,0)] = 3! \cdot [\overline{\mathcal{M}}_{1,1}]$. From Theorem 2.2.21 we now deduce:

Proposition 3.2.2. With the notation of Paragraph 3.2.1. If $h \geq 1$ we have

$\xi_A^* \phi'_{n*}[\mathcal{A}\!\mathit{dm}(g,h)]$

$= \binom{2g+2-4h}{3} \pi_{B_1*} \phi_{f_1*}$ [] $+ \pi_{B_2*} \phi_{f_2*}$ []

$$= \phi_{n+1*}[\mathcal{A}\!\mathit{dm}(g-1,h)_{2m}] \otimes [\overline{\mathcal{M}}_{1,1}] + \pi_{B_2*}\phi_{f_2*}[\mathcal{A}\!\mathit{dm}(g-2,h-1)_{2m+2} \times \overline{\mathcal{M}}^D_{1,1}]$$

If $h = 0$ the term with the diagonal disappears, in other words the above expression simplifies to

$$\xi_{A_1}^* \phi_{n*}([\mathcal{A}\!\mathit{dm}(g,0)]) = \phi_{n+1*}[\mathcal{A}\!\mathit{dm}(g-1,0)_{2m}] \otimes \phi_{1*}[\mathcal{A}\!\mathit{dm}(1,0)].$$

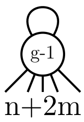
Remark 3.2.3. It follows immediately from Proposition 1.3.11 that

$$[\overline{\mathcal{M}}^D_{1,1}] = 2 \cdot [\overline{\mathcal{M}}_{1,1}] \otimes \Bigg[\quad \Bigg] + 2 \cdot \Bigg[\quad \Bigg] \otimes [\overline{\mathcal{M}}_{1,1}]$$

$$\in A^1(\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1}) = A^1(\overline{\mathcal{M}}_{1,1}) \otimes A^1(\overline{\mathcal{M}}_{1,1}).$$

Therefore if expressions for $\phi_{n*}[\mathcal{Adm}(g,h)_{2m}]$ and $\phi_{n*}[\mathcal{Adm}(g-2,h-1)_{2m+2}]$ are known we obtain one for $\phi_{n+1}[\mathcal{Adm}(g-1,h)_{2m}]$.

3.2.4. Let

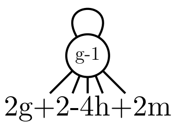
$A =$ 

.

We will identify the diagram

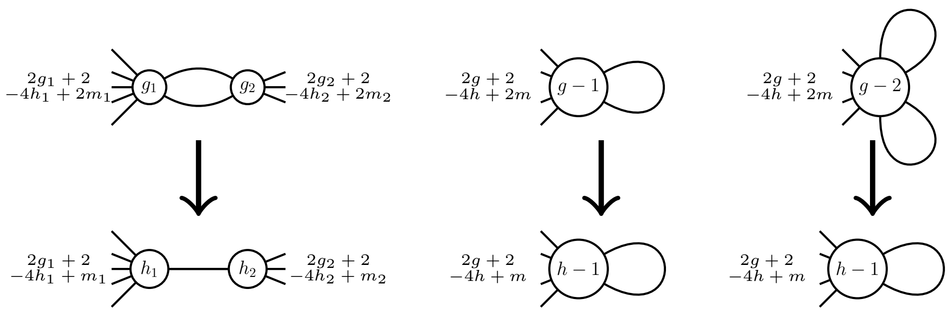
$$\begin{tikzcd}
\coprod_{B\in\mathfrak{A}} \coprod_{(\Gamma,\tau,f)\in\mathfrak{H}_B} \mathcal{Adm}_{(\Gamma,\tau)} \arrow[r, "\eta_\Gamma"] \arrow[d, "\phi_f"] & \mathcal{Adm}(g,h) \arrow[d, "\phi"] \\
\coprod_{B\in\mathfrak{A}} \overline{\mathcal{M}}_B \arrow[r, "\pi_B"] \arrow[d, "\pi_B"] & \overline{\mathcal{M}}_{g,2g+2-4h+2m} \arrow[d, "\pi^{(k)}"] \\
\overline{\mathcal{M}}_A \arrow[r, "\xi_A"] & \overline{\mathcal{M}}_{g,n+2m}
\end{tikzcd}$$

There is only one graph in $\mathfrak{A}$. It consists of attaching the remaining legs to the single vertex of A, i.e. $\mathfrak{A}$ consists of the single graph

$B =$ 

The graph B admits the following types of admissible pairs.

$(\Gamma,\tau) =$ $\qquad$ $(\Gamma_1,\tau_1) =$ $\qquad$ $(\Gamma_2,\tau_2) =$

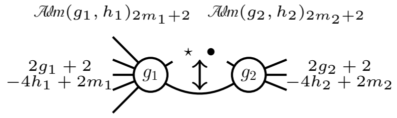


.

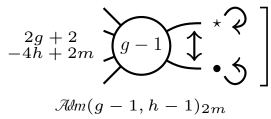
There are 2 isomorphism classes of B-structures on admissible pairs of the form (Γ,τ) and there are $\binom{2g+2-4h}{2g_1+2-4h_1}\binom{m}{m_1}$ ways to distribute the nodes. There is 1 isomorphism class of B-structures on (Γ_1,τ_1). Likewise there is 1 isomorphism class of B-structures on (Γ_2,τ_2). From Theorem 2.2.21 we now obtain:

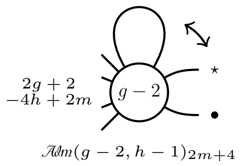
Proposition 3.2.5. With the notation of Paragraph 3.2.4 we have

$$\xi_A^* \phi_{n*}[\mathcal{A}dm(g,h)_{2m}]$$

$$= \sum_{\substack{g_1+g_2=g-1\\ h_1+h_2=h\\ m_1+m_2=m}} 2\binom{2g+2-4h}{2g_1+2-4h_1}\binom{m}{m_1}\pi_{B*}\phi_{f*}\Big[$$


$$\Big] \qquad (\dagger)$$

$$+\,\pi_B\phi_{f_1*}\Big[$$


$$\Big] + 2\pi_B\phi_{f_1*}\Big[$$


$$\Big]$$

$$= \sum_{\substack{g_1+g_2=g-1\\ h_1+h_2=h\\ m_1+m_2=m}} 2\binom{2g+2-4h+m}{2g_1+2-4h_1+m_1}\phi_{n*}[\mathcal{A}dm(g_1,h_1)_{2m_1+2}\times \mathcal{A}dm(g_2,h_2)_{2m_2+2}] \qquad (\dagger)$$

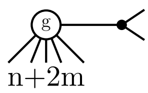
$$+\,\phi_{n+2*}[\mathcal{A}dm(g-1,h-1)_{2m}] + \xi_*^{\mathrm{irr}}\phi_{n*}[\mathcal{A}dm(g-2,h-1)_{2m+4}]$$

where in the second line $\bullet$ and $\star$ indicate the points glued together by the morphism ξ_A and where $\xi_*^{\mathrm{irr}}\colon \overline{\mathcal{M}}_{g-2,2g+2-4h+2m+4} \to \overline{\mathcal{M}}_{g-1,2g+2+2m+2}$ is the morphism gluing the last two marked points together. Note that if $h=0$ then terms on the last line disappear.

Remark 3.2.6. Note that $\phi_{n*}[\mathcal{A}dm(g-1,h)_{2m}]$ is one of the components of the sum (†).

Remark 3.2.7. The pullbacks performed in Proposition 3.2.2 and 3.2.5 are not the only possible pullbacks we can do. However they will cover most of the cases we are interested in. We introduce one useful pullback.

Proposition 3.2.8. Let

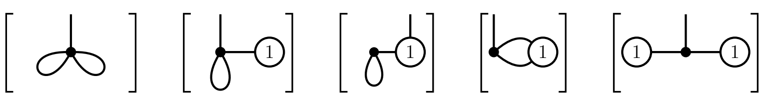
$$A =$$


be the graph. We have $\xi_A^*\phi_{n*}[\mathcal{A}dm(g,h)_{2m+2}] = \phi_{n+1*}[\mathcal{A}dm(g,h)_{2m}]$.

Proof. Follows from Theorem 2.2.21. The computation is straightforward. □

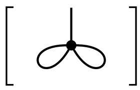
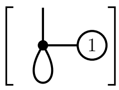
The Class of $\overline{\mathcal{B}}_{2,1,0}$

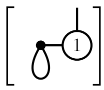
As a first example of the above methods we will calculate the class of the locus $\overline{\mathcal{B}}_{2,1,0}$ in $A^2(\overline{\mathcal{M}}_{2,1})$. The bielliptic locus $\overline{\mathcal{B}}_{2,1,0}$ has codimension 2 inside $\overline{\mathcal{M}}_{2,1}$, indeed $\dim \mathcal{A}dm(2,1) = \dim \overline{\mathcal{M}}_{1,2} = 2$ while $\dim \overline{\mathcal{M}}_{2,1} = 4$. A basis for $A^2(\overline{\mathcal{M}}_{2,1})$ is given by the tautological classes (see [Fab88])

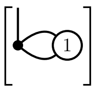


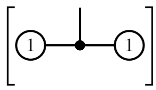
(3.4)

Proposition 3.2.9. The class of the bielliptic locus $\overline{\mathcal{B}}_{2,1,0}$ in $A^2(\overline{\mathcal{M}}_{2,1})$ is given by

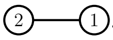
$$[\overline{\mathcal{B}}_{2,1,0}] = \frac{1}{4}$$

$$+\frac{7}{2}$$


$$-\frac{1}{2}$$


$$+6$$


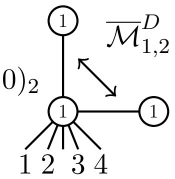
$$+24$$


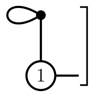
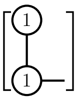
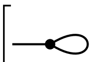
Proof. Let

$$A =$$  .

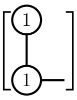
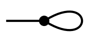
By Proposition 3.2.2 we then have

$$\xi_A^*[\overline{\mathcal{B}}_3] = \frac{1}{4!}\xi_A^*\phi_{n*}([\mathcal{Adm}(3,1)])$$

$$= \frac{1}{4!}\Bigg(\pi_{B_1*}\phi_{f_1*}[\mathcal{Adm}(2,1)\times\mathcal{Adm}(1,0)] + \pi_{B_2}\phi_{f_2*}\Bigg[\mathcal{Adm}(1,0)_2$$  $$\Bigg]\Bigg)$$

$$= \phi_{1*}[\mathcal{Adm}(2,1)]\otimes[\overline{\mathcal{M}}_{1,1}] + 2\Bigg[$$  $$\Bigg]\otimes[\overline{\mathcal{M}}_{1,1}] + 2\Bigg[$$  $$\Bigg]\otimes\Bigg[$$  $$\Bigg]$$

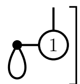
Note that the class

$$2\Bigg[$$  $$\Bigg]\otimes\Bigg[$$  $$\Bigg]$$

lies in $A^1(\overline{\mathcal{M}}_{2,1})\otimes A^1(\overline{\mathcal{M}}_{1,1})$ while the others lie in $A^2(\overline{\mathcal{M}}_{2,1})\otimes A^0(\overline{\mathcal{M}}_{1,1})$. Therefore composing with the projection:

$$P\colon A^2(\overline{\mathcal{M}}_{2,1}\times\overline{\mathcal{M}}_{1,1})\to A^2(\overline{\mathcal{M}}_{2,1})\otimes A^0(\overline{\mathcal{M}}_{1,1})$$

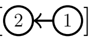
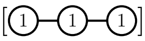
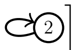
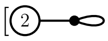
we get

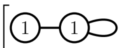
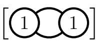
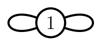
$$P\xi_A^*([\overline{\mathcal{B}}_3])) = [\overline{\mathcal{B}}_{2,1,0}]\otimes[\overline{\mathcal{M}}_{1,1}] + 2\Bigg[$$  $$\Bigg].$$

The class $[\overline{\mathcal{B}}_3]$ has already been calculated in [FP15][1] to be:

$$\frac{1917}{2}\lambda^2 - \frac{369}{2}\lambda\delta_0 - 417\lambda\delta_1 + 9\delta_0^2 + \frac{87}{2}\delta_0\delta_1 + \frac{105}{2}\delta_1^2 - \frac{9}{2}\kappa_2 \tag{3.5}$$

After a change of basis to decorated stratum classes we have:

$$[\overline{\mathcal{B}}_3] = 30\,[$$  $$] - 6\,[$$  $$] + \frac{15}{4}\Big[$$  $$\Big] - \frac{15}{2}\,[$$  $$] \tag{3.6}$$

$$+\frac{3}{2}\Big[$$  $$\Big] + \frac{45}{4}\,[$$  $$] + \frac{1}{8}\Big[$$  $$\Big]$$

Where the change of basis matrix is given by:

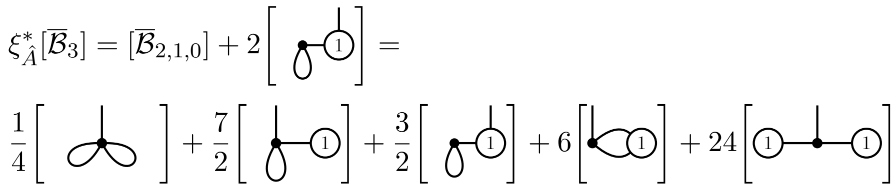
$$\begin{pmatrix} \frac{2}{21} & 0 & 0 & 0 & 0 & -1 & \frac{41}{21} \\ \frac{2}{35} & 0 & \frac{2}{5} & 0 & 0 & 2 & -\frac{8}{35} \\ \frac{1}{42} & 0 & 0 & -2 & 0 & 0 & \frac{5}{21} \\ -\frac{1}{126} & 0 & \frac{1}{12} & 0 & 1 & -\frac{1}{12} & -\frac{41}{252} \\ \frac{4}{105} & \frac{1}{5} & \frac{1}{10} & 0 & 1 & 0 & -\frac{2}{105} \\ \frac{1}{14} & \frac{2}{5} & 0 & 2 & 0 & 0 & \frac{11}{35} \\ \frac{5}{252} & \frac{1}{5} & 0 & 2 & 0 & 0 & -\frac{1}{630} \end{pmatrix}.$$

[1] There is a mistake in the published version of the paper. The class we write down is the corrected version of their class. See Remark 3.4.2 for a more careful analysis of their mistake.

On the other hand the composition $P \circ \xi_{\hat{A}}^* \colon A^2(\overline{\mathcal{M}}_3) \to A^2(\overline{\mathcal{M}}_{2,1} \times \overline{\mathcal{M}}_{1,1}) \to A^2(\overline{\mathcal{M}}_{2,1}) \otimes A^0(\overline{\mathcal{M}}_{0,3})$ can be computed using Theorem 2.1.30 (and either a tedious calculation by hand or a computer program such as [Sch]). In terms of the bases of (3.4) and (3.6) it is given by the matrix

$$\begin{pmatrix} -\frac{1}{120} & 0 & \frac{1}{6} & 0 & -\frac{1}{6} & 0 & 1 \\ -\frac{1}{40} & -\frac{1}{12} & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{120} & 0 & 1 & 1 & 0 & 0 & 0 \\ -\frac{1}{5} & 0 & 1 & 0 & -2 & 1 & 0 \\ \frac{3}{5} & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

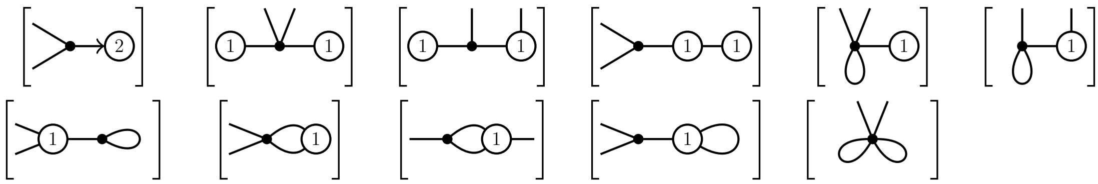
The resulting equality is



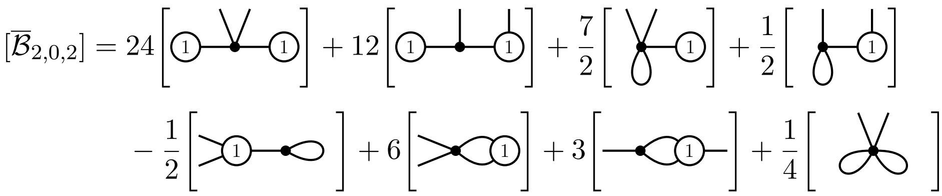
□

The Class of $\overline{\mathcal{B}}_{2,0,2}$.

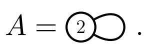
We will now compute the class of $[\overline{\mathcal{B}}_{2,0,2}]$ using Proposition 3.2.5 and an expression in terms of decorated stratum classes of $[\overline{\mathcal{B}}_3]$. A basis for $A^2(\overline{\mathcal{M}}_{2,2})^{\mathfrak{S}_2}$ is given by (see [Get98b, Section 9])



Proposition 3.2.10. In $A^2(\overline{\mathcal{M}}_{2,2})^{\mathfrak{S}_2}$ we have:

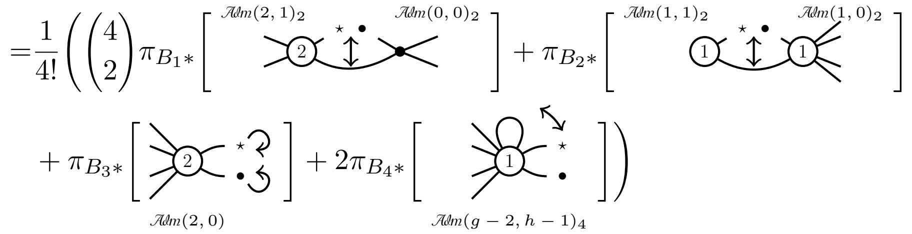


Proof. Consider the graph



By Proposition 3.2.5 we have

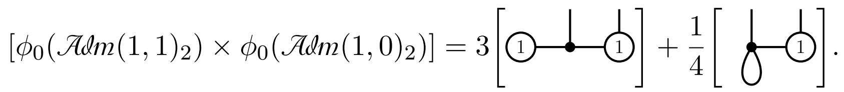
$$\xi_A^*([\overline{\mathcal{B}}_3]) = \frac{1}{4!}\xi_A^*\phi_{0*}[\mathcal{A}dm(3,1)]$$



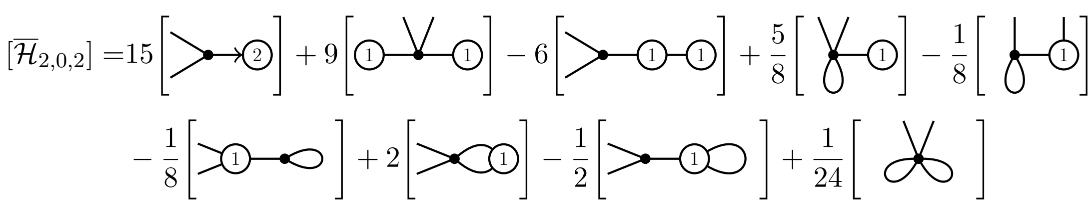
$$\begin{aligned}&= 2[\phi_0(\mathcal{A}dm(2,1)_2)] + 2[\phi_0(\mathcal{A}dm(1,1)_2) \times \phi_0(\mathcal{A}dm(1,0)_2)]\\ &\quad + 2[\phi_2(\mathcal{A}dm(2,0))] + 2[\xi^{\mathrm{irr}}\phi_0(\mathcal{A}dm(1,0)_4)]\end{aligned}$$

The class $[\phi_0(\mathcal{A}dm(1,1)_2)] =: [\overline{\mathcal{B}}_{1,0,2}]$ equals the class $[\overline{\mathcal{H}}_{1,2,0}]$ which has for example been computed in [Pag13, Theorem 3.33]. Indeed every smooth genus 1 curve admits a bielliptic involution τ. Let P and Q be two marked points switched by τ. Let σ be the hyperelliptic involution interchanging P and Q. The involution $\tau \circ \sigma$ is a hyperelliptic involution fixing P and Q. We have constructed an inclusion $\overline{\mathcal{B}}_{1,0,2} \hookrightarrow \overline{\mathcal{H}}_{1,2,0}$. Since both loci are complete irreducible and of dimension 1 equality follows.

We have $[\phi_0(\mathcal{A}dm(1,0)_2)] = [\overline{\mathcal{H}}_{1,0,2}] = [\overline{\mathcal{M}}_{1,2}]$, so

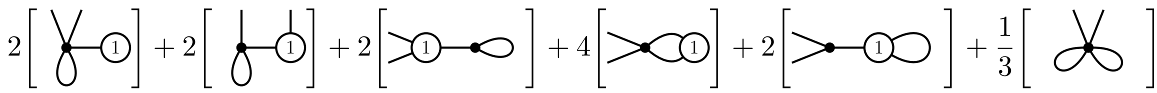


The class $[\phi_2(\mathcal{A}dm(2,0)] = \overline{\mathcal{H}}_{2,2,0}$ is computed in [Tar15, Theorem 1] (under the notation $[\mathcal{DR}_2(2)])^2$; in terms of our basis it is



We have[3]

$$[\xi^{\mathrm{irr}}\phi_{0*}(\mathcal{A}dm(1,0)_4)] = [\xi^{\mathrm{irr}}(\overline{\mathcal{H}}_{1,0,4})] =$$



The pullback map $\xi_A^*\colon A^2(\overline{\mathcal{M}}_3) \to A^2(\overline{\mathcal{M}}_{2,2})^{\mathfrak{S}_2}$ can again be determined by Theorem 2.1.30.

[2] Alternatively this class can be computed in terms of decorated stratum classes by using Proposition 3.2.5 and an expression in terms of decorated stratum classes of $[\overline{\mathcal{H}}_3]$.

[3] The author does not know of a reference for this, however the computation can be made using Proposition 3.2.5 and an expression in terms of decorated stratum classes of $\overline{\mathcal{H}}_{2,0,2}$.

We have

$$\xi_A^* = \begin{pmatrix} 2 & 0 & -8 & 0 & 0 & 0 & 0 \\ 4 & 2 & -8 & 0 & -8 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & -2 & 0 \\ 0 & 2 & \frac{8}{5} & 0 & -4 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{7}{5} & 0 & \frac{5}{3} & 0 & -4 \\ 0 & 0 & \frac{19}{10} & 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{31}{15} & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 2 & -8 \\ 0 & 0 & \frac{5}{3} & 0 & 0 & 0 & -2 \\ 0 & 0 & \frac{2}{15} & 0 & 2 & 0 & -4 \\ 0 & 0 & \frac{13}{45} & 0 & 0 & 0 & \frac{4}{3} \end{pmatrix}$$

in terms of the basis for $A^2(\overline{\mathcal{M}}_3)$ given in the proof of 3.2.9 and the basis of $A^2(\overline{\mathcal{M}}_{2,2})^{\mathfrak{S}_2}$ given at the beginning of this subsection. The statement of this Proposition follows.

□

Remark 3.2.11. We could also use this expression of $[\overline{\mathcal{B}}_{2,0,2}]$ and Proposition 3.2.8 to compute the class $[\overline{\mathcal{B}}_{2,1,0}]$.

3.3 Repeated Pullback to Lower Genus

We can generalize the method of Section 3.1 by intersecting with (decorated) boundary strata which are not necessarily of complementary codimension. For every stable graph A we again form the fiber diagram

$$\begin{tikzcd} \coprod_{B\in\mathfrak{A}} \coprod_{(\Gamma,\tau,f)\in\mathfrak{H}_B} \mathcal{A}dm_{(\Gamma,\tau)} \arrow[r, "\eta_\Gamma"] \arrow[d, "\phi_f"] & \mathcal{A}dm(g,h) \arrow[d, "\phi"] \\ \coprod_{B\in\mathfrak{A}} \overline{\mathcal{M}}_B \arrow[r, "\pi_B"] \arrow[d, "\pi_B"] & \overline{\mathcal{M}}_{g,2g+2-4h+2m} \arrow[d, "\pi^{(k)}"] \\ \overline{\mathcal{M}}_A \arrow[r, "\xi_A"] & \overline{\mathcal{M}}_{g,n+2m} \end{tikzcd} .$$

Theorem 2.2.21 expresses the pullback $\xi_A^*\phi_{n*}([\mathcal{A}dm(g,h)_{2m}])$ as a sum of classes of the form $\pi_{B*}\phi_{f*}([\mathcal{A}dm_{(\Gamma,\tau)}])$. It is often the case that for a particular stable graph A we can compute all terms of this form in terms of decorated stratum classes.

Suppose that for a boundary stratum A we can determine $\xi_A^*\phi_{n*}([\mathcal{A}dm(g,h)_{2m}])$ in terms of decorated stratum classes. Using Theorem 2.1.30 we can compute $\xi_A^*\colon R^\bullet(\overline{\mathcal{M}}_{g,n}) \to R^\bullet(\overline{\mathcal{M}}_A)$ explicitly as a linear map in terms of (generating sets of) decorated stratum classes. Taking the inverse image of $\xi_A^*\phi_{n*}([\mathcal{A}dm(g,h)_{2m}])$ under ξ_A^* then determines the class $\phi_{n*}([\mathcal{A}dm(g,h)_{2m}])$ in terms of decorated stratum classes up to the kernel of ξ_A^*.

If we can do this for a number of graphs A_i and

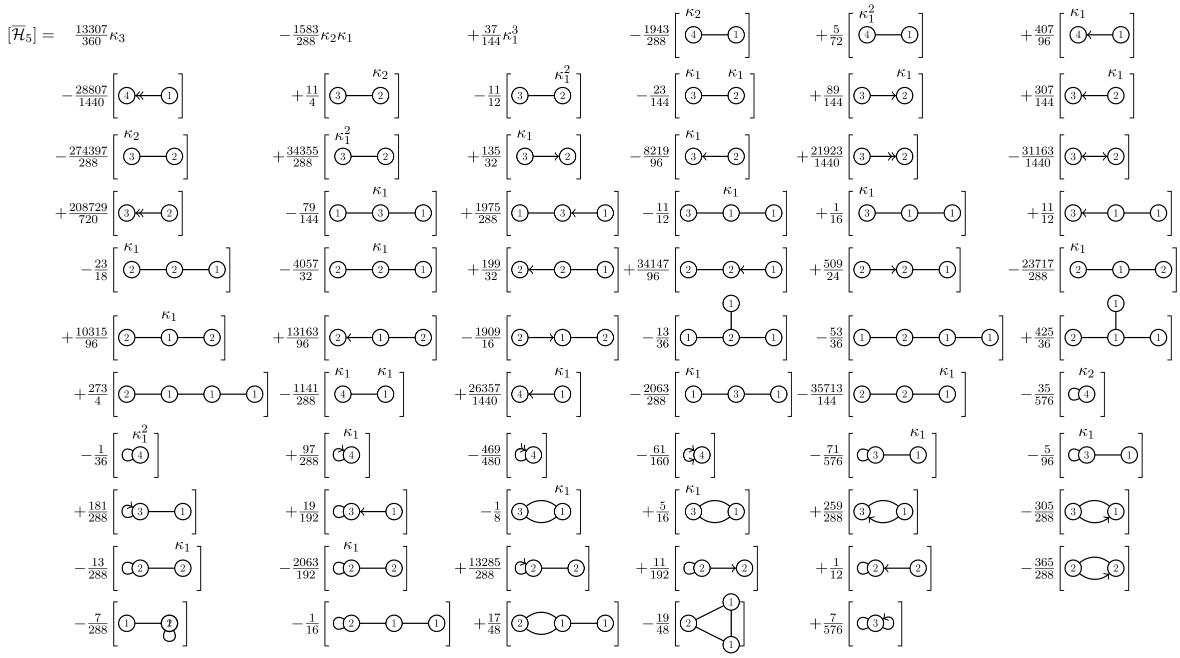
$$\bigcap \ker \xi_{A_i}^* = \{0\}$$

then we can completely determine the class $\phi_{n*}([\mathcal{A}dm(g,h)_{2m}])$.

Remark 3.3.1. The perfect pairing method of Section 3.1 can be seen as a special case of this where we take *decorated* graphs of codimension equal to the dimension of $\phi_n[\mathcal{A}dm(g,h)_{2m}]$.

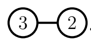
The Class of $\overline{\mathcal{H}}_5$

Theorem 3.3.2. In $R^3(\overline{\mathcal{M}}_5)$ we have

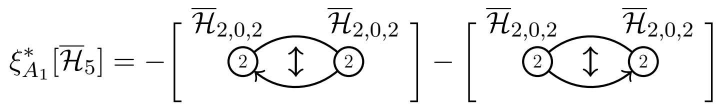


Remark 3.3.3. In all other statements in this thesis where we express a class $[\phi_n\,\mathcal{A}dm(g,h)_{2m}]$ in terms of decorated stratum classes the expression will be in terms of a *basis*. The expression of Theorem 3.3.2 is in terms of a *generating set* determined up to the extended Faber-Zagier relations (computed using [Pix]). It is conjectured that the extended Faber-Zagier relations are all relations on $R^\bullet(\overline{\mathcal{M}}_{g,n})$ but the author is unaware of any proof of this in the case of $R^3(\overline{\mathcal{M}}_5)$.

Proof of Theorem 3.3.2. Consider the following two graphs

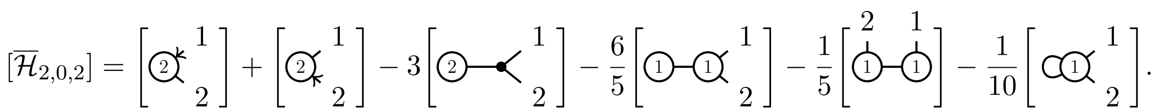
$A_1 =$  $\qquad A_2 =$ .

By Theorem 2.2.21 we have



$$= -\psi_1 \cap [\overline{\mathcal{H}}_{2,0,2}] \otimes [\overline{\mathcal{H}}_{2,0,2}] - [\overline{\mathcal{H}}_{2,0,2}] \otimes \psi_1 \cap [\overline{\mathcal{H}}_{2,0,2}] \in R^\bullet(\overline{\mathcal{M}}_{2,2} \times \overline{\mathcal{M}}_{2,2}).$$

The class $[\overline{\mathcal{H}}_{2,0,2}] \in A^1(\overline{\mathcal{M}}_{2,2})$ was originally computed in [BP00, Lemma 6]. We have:

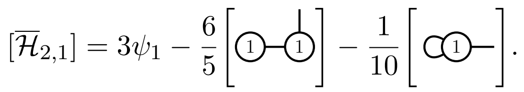


This immediately gives $\psi_1 \cap [\overline{\mathcal{H}}_{2,0,2}]$ as a class in terms of decorated stratum classes. We thus know $\xi^*_{A_1}([\overline{\mathcal{H}}_5])$ in terms of decorated stratum classes.

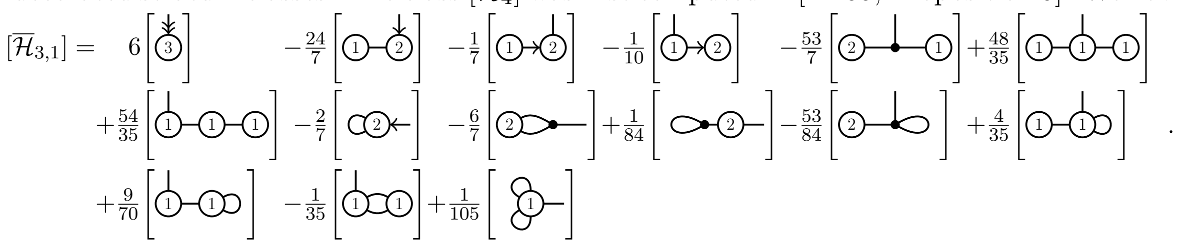
By Theorem 2.2.21 we have

$$\xi^*_{A_2}([\overline{\mathcal{H}}_5]) = [\overline{\mathcal{H}}_{3,1} \times \overline{\mathcal{H}}_{2,1}] \in H^6(\overline{\mathcal{M}}_{3,1} \times \overline{\mathcal{M}}_{2,1})$$

The class $[\overline{\mathcal{H}}_{2,1,0}]$ has originally been computed in [EH87, Theorem 2.2]. We have



The class $[\overline{\mathcal{H}}_{3,1}]$ can be computed using Proposition 3.2.2 and an expression for $[\overline{\mathcal{H}}_4]$ in terms of decorated stratum classes. The class $[\overline{\mathcal{H}}_4]$ was first computed in [FP05, Proposition 5]. We have



□

3.4 Intersecting with the Hyperelliptic Locus

In [FP15] the authors calculate the class $[\overline{\mathcal{B}}_3]$ by pulling back along the hyperelliptic map $\phi_0\colon \mathcal{A}dm(3,0) \to \overline{\mathcal{M}}_3$ and then pushing forward along the target map $\mathcal{A}dm(3,0) \to \overline{\mathcal{M}}_{0,8}$. The space $\overline{\mathcal{M}}_{0,8}$ is relatively easy to understand; the inverse image of $\overline{\mathcal{B}}_3$ along ϕ_0 can be determined by set theoretic arguments. This allows the authors to compute the class $I := \rho_*\phi_0^*([\overline{\mathcal{B}}_3])$ up to a constant. In [FP15] the composition $M := \rho_* \circ \phi_0^*\colon A^2(\overline{\mathcal{M}}_3) \to A^2(\mathcal{A}dm(3,0)) \to A^2(\overline{\mathcal{M}}_{0,8})$, in terms of a basis for $A^2(\overline{\mathcal{M}}_3)$ consisting of products of divisors and κ_2, is determined by computing the pull-push $\rho_*\phi_0^*(D)$ for all boundary divisors $D \in A^1(\overline{\mathcal{M}}_3)$ and multiplying out on both sides, and by directly computing $\rho_*\phi_0^*(\kappa_2)$. Since $\dim A^2(\overline{\mathcal{M}}_{0,8}) = 6$ this determines 5 out of the 7 coefficients of the class $[\overline{\mathcal{B}}_3]$.

Our methods allow us to do the same trick for bielliptic classes. The composition

$$\rho_*\phi_n^* : \overline{\mathcal{M}}_{g,n+2m} \to \mathcal{A}dm(g,0)_{2m} \to \overline{\mathcal{M}}_{0,2g+2+m}$$

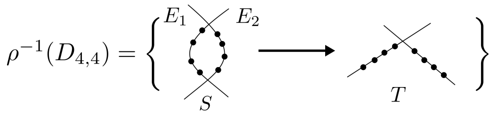
as a map of vector spaces can be computed using Theorem 2.2.26. We shall analyze the set theoretic inverse image $\phi^{-1}([\overline{\mathcal{B}}_{g,n,2m}])$ for the classes $[\overline{\mathcal{B}}_{2,1,0}]$ and $[\overline{\mathcal{B}}_4]$. Using this and the method of Section 3.3 we shall compute these classes in terms of a basis of decorated stratum classes.

Remark 3.4.1. The classes we compute could in principle have been computed using exclusively the methods introduced earlier in this chapter. The author does not know whether the method of this section allows us to compute strictly more than we did before.

Remark 3.4.2. A mistake is made in Proposition 3 of [FP15] where the set $\rho(\phi_0^{-1}(\overline{\mathcal{B}}_3))$ is analyzed. Let $I_8^{\mathrm{inv}} \subset \overline{\mathcal{M}}_{0,8}$ be the closure of the locus of smooth curves admitting an involution which sends marked points to marked points and does not fix any of the marked points. Note that I_8^{inv} is the orbit of $\overline{\mathcal{H}}_{0,0,8}$ under the action of the symmetric group. Proposition 3 of [FP15] claims the equality

$$\rho\phi_0^{-1}(\overline{\mathcal{B}}_3) = I_8^{\mathrm{inv}}.$$

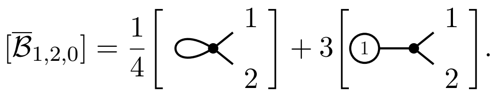
Which is false. In particular Claim 1 in the proof of Proposition 3 is not true: let $D_{4,4} \subset \overline{\mathcal{M}}_{0,8}$ be the locus of curves T with at least one (separating) node P and four marked points on either side of P. The inverse image $\rho^{-1}(D_{4,4})$ consists of admissible covers whose source curve S consists of two genus 1 curves E_1, E_2 glued together in two points:



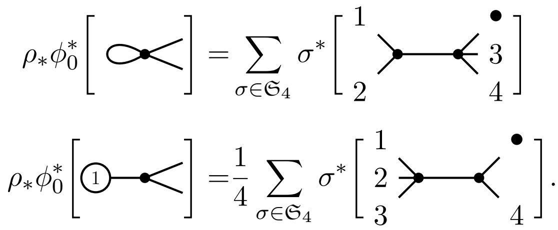
The admissible cover $S \to T$ lies in $\phi_0^{-1}(\overline{\mathcal{B}}_3)$ if S admits a bielliptic involution. Suppose E_1 admits a bielliptic involution switching the two nodes. A bielliptic involution on the whole of S is then given by taking the bielliptic involution on E_1 and the restriction of the hyperelliptic involution on E_2. This component then corresponds to the pullback of $[\overline{\mathcal{B}}_{1,0,2} \times \overline{\mathcal{H}}_{1,0,2}]$ under the map

$$\mathcal{Adm}(1,0)_2 \times \mathcal{Adm}(1,0)_2 \to \overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{1,2}.$$

To compute this pullback we have to compute the pullback of $[\overline{\mathcal{B}}_{1,0,2}]$ under $\phi_\Gamma\colon \mathcal{Adm}(1,0)_2 \to \overline{\mathcal{M}}_{1,2}$. By Proposition 3.2.5 and using that $[\overline{\mathcal{B}}_2] = \frac{3}{2}\delta_0 + 6\delta_1$ ([FP15, Proposition 2]) it can easily be determined that

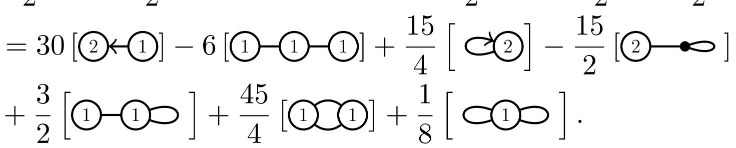


Using Theorem 2.2.26 one can compute that



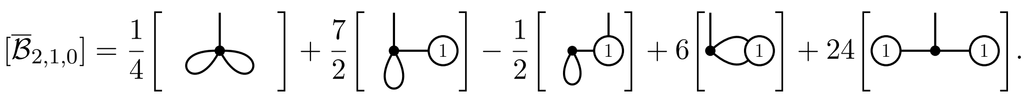
The missing factor in $\rho_*\phi_0^*([\overline{\mathcal{B}}_3])$ in [FP15] is therefore given by $\frac{1}{4}d_{4,2,2} + \frac{3}{4}d_{4,1,3}$ (in the notation of 2.2.28). The resulting correct statement of [FP15, Theorem 1] is

$$[\overline{\mathcal{B}}_3] = \frac{1917}{2}\lambda^2 - \frac{369}{2}\lambda\delta_0 - 417\lambda\delta_1 + 9\delta_0^2 + \frac{87}{2}\delta_0\delta_1 + \frac{105}{2}\delta_1^2 - \frac{9}{2}\kappa_2$$



The Class of $\overline{\mathcal{B}}_{2,1,0}$

As an easy example we will start by proving Theorem 3.2.9 by pulling back $[\overline{\mathcal{B}}_{2,1,0}]$ along the map $\phi_1\colon \mathcal{Adm}(2,0) \to \overline{\mathcal{M}}_{2,1}$ (and then pushing it forward to the target space $\overline{\mathcal{M}}_{0,6}$). In other words we will prove again that



We therefore want to compute the pull-push

$$\rho_* \phi_1^* \colon A^2(\overline{\mathcal{M}}_{2,1}) \to A^2(\mathcal{A}\mathcal{d}m(2,0)) \to A^2(\overline{\mathcal{M}}_{0,6})$$

in terms of a basis of decorated stratum classes of $A^2(\overline{\mathcal{M}}_{2,1})$ and of the class $[\overline{\mathcal{B}}_{2,1,0}]$. Note that the image of $\rho_* \phi_1^*$ lies in the $\mathfrak{S}_5$-invariant part $A^2(\overline{\mathcal{M}}_{0,8})^{\mathfrak{S}_5}$.

Notation 3.4.3. Let $D_{i\bullet,j,k}$, $D_{i,j\bullet,k}$ be the loci in $\overline{\mathcal{M}}_{0,6}$ corresponding to the sum of all graphs with set of legs $L = \{1, ..., 5, \bullet\}$ with three vertices of genus 0, an edge from v_1 to v_2 and from v_2 to v_3, such that $\zeta(\bullet) = v_1$ respectively $\zeta(\bullet) = v_2$ and v_1, v_2, v_3 have i, j and k of the remaining legs on them. Let $d_{i\bullet,j,k}$ and $d_{i,j\bullet,k}$ be the corresponding elements of $A^2(\overline{\mathcal{M}}_{0,6})^{\mathfrak{S}_5}$ (see Notation 2.2.28).

A set of generators of $A^2(\overline{\mathcal{M}}_{0,6})^{\mathfrak{S}_5}$ is then given by:

$$d_{3,1,1\bullet} \quad d_{3,\bullet,2} \quad d_{2,1\bullet,2} \quad d_{2\bullet,1,2} \quad d_{2,2,1\bullet}\ .$$

The fundamental relation in $A^\bullet(\overline{\mathcal{M}}_{0,4})$ (see Example 1.3.7) pulls back to the following 2 relations

$$\begin{aligned} d_{3,1,1\bullet} &= 2d_{3,\bullet,2} \\ d_{2,1\bullet,2} &= \frac{1}{2}(-3d_{3,\bullet,2} + d_{2,2,1\bullet} + d_{2\bullet,1,2}). \end{aligned}$$

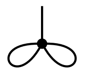
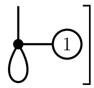
A basis can be chosen to be

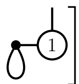
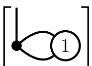
$$d_{3,\bullet,2} \quad d_{2\bullet,1,2} \quad d_{2,2,1\bullet}\ . \tag{3.7}$$

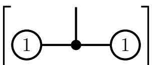
Proposition 3.4.4. In terms of the ordered basis of (3.4) and (3.7) the composition $\rho_* \circ \phi_1^* \colon A^2(\overline{\mathcal{M}}_{2,1}) \to A^2(\overline{\mathcal{M}}_{0,6})^{\mathfrak{S}_5}$ is given by the matrix

$$\begin{pmatrix} -3 & \frac{3}{2} & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 0 \\ 3 & 0 & 0 & -\frac{1}{12} & 0 \end{pmatrix}$$

Proof. Using Theorem 2.2.26 it is relatively easy to compute that

$\phi_1^*\Big[$  $\Big] = 2d_{2,1\bullet,2} + 2d_{2,2,1\bullet}$ $\qquad$ $\phi_1^*\Big[$  $\Big] = \frac{1}{2}d_{3,\bullet,2} + \frac{1}{2}d_{3,1,1\bullet}$

$\phi_1^*\Big[$  $\Big] = \frac{1}{2}d_{2\bullet,1,2}$ $\qquad$ $\phi_1^*\Big[$  $\Big] = -\frac{1}{12}d_{2,2,1\bullet} - \frac{1}{4}d_{3,1,1\bullet}$

$\phi_1^*\Big[$  $\Big] = 0.$

$\square$

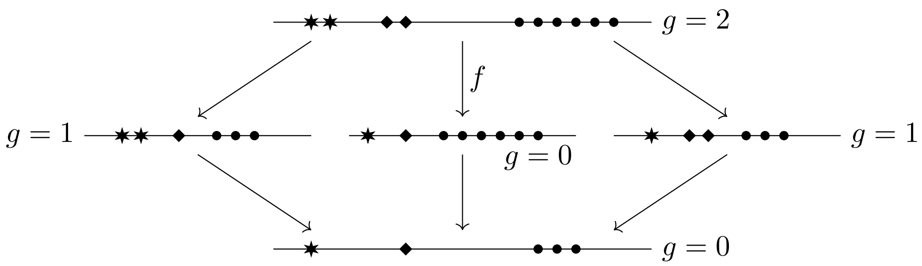
Proposition 3.4.5. We have

$$\rho_* \phi_1^*([\overline{\mathcal{B}}_{2,1,0}]) = \frac{1}{4}d_{2,2,1\bullet} + \frac{3}{4}d_{3,1,1\bullet}.$$

Proof. We will first consider the inverse image $\phi_1^{-1}(\overline{\mathcal{B}}_{2,1,0})$ and its image $\rho\phi_1^{-1}(\overline{\mathcal{B}}_{2,1,0})$ on the target space. First note that the locus $\overline{\mathcal{B}}_{2,1,0}$ is of codimension 2 in $\overline{\mathcal{M}}_{2,1}$. The inverse image of $\overline{\mathcal{B}}_{2,1,0}$ is therefore of codimension at most 2 (or empty). Also note that $\rho(\phi_1^{-1}(\overline{\mathcal{B}}_{2,1,0}))$ is invariant under the action of $\mathfrak{S}_5$. The space $\mathcal{A}dm(2,0)$ has a stratification induced by the stratification of $\overline{\mathcal{M}}_{0,6}$; we will argue by analyzing the different $\mathfrak{S}_5$-invariant locally closed strata up to codimension 2.

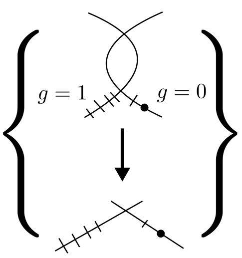
Codimension 0. Let $f\colon S \to T$ be an admissible hyperelliptic cover of smooth genus 2 curves with hyperelliptic involution σ. If S also admits a bielliptic involution τ then the involutions σ and τ commute. The involution $\sigma \circ \tau$ is a bielliptic involution. The involution τ induces an involution on T. Modding out by this involution gives a map $T \to R$ where R is a smooth genus 0 curve. By the Riemann-Hurwitz formula, the ramification divisor of the composition $g\colon S \to T \to R$ has degree 10. The only ramification points of g are the fixed points of σ, τ and $\sigma \circ \tau$. Therefore the fixed points of σ and of τ and of $\sigma \circ \tau$ are distinct.

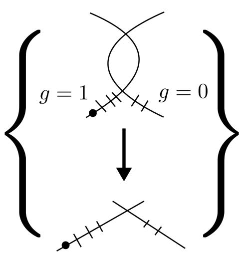
The situation is be summarized by the following diagram. Where the upper middle map is the hyperelliptic map. The lower middle map is the map $T \to R$ and the upper maps on the right and left are the two bielliptic maps.

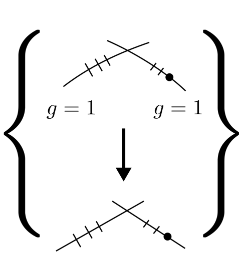


Therefore the restriction of $\phi_1^{-1}(\overline{\mathcal{B}}_{2,1,0})$ to the smooth locus is empty.

Codimension 1. Let us then look at the strata of $\mathcal{A}dm(2,0)$ given by the $\mathfrak{S}_5$-invariant codimension 1 strata of $\overline{\mathcal{M}}_{0,6}$. There are three such strata:

$D_{4,1\bullet} =$ 

$D_{3\bullet,2} =$ 

$D_{3,2\bullet} =$ 

The locus $D_{4,1\bullet}$ has the bielliptic locus as a codimension 1 sublocus. It is given by taking the bielliptic involution, switching the nodes, on the genus 1 component on the left, while taking the hyperelliptic involution on the component on the right (similarly to Remark 3.4.2). The other two components do not contain any bielliptic curves fixing $\bullet$ as a codimension 1 sublocus. Indeed a bielliptic involution on a curve of genus 1 cannot fix any points excluding both cases.

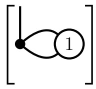
Codimension 2. The codimension two strata are given by:

$$D_{3,1,1\bullet} \quad D_{3,\bullet,2} \quad D_{2\bullet,1,2} \quad D_{2,2,1\bullet} \quad D_{2,1\bullet,2}$$

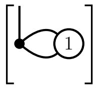
Using similar arguments it can be seen that none of these strata generically admit a bielliptic involution.

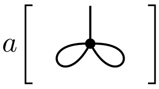
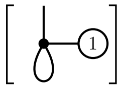
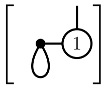
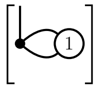
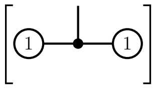
In the same way as in Remark 3.4.2 it can be seen that the locus of curves inside $D_{4,1\bullet}$ admitting a bielliptic involution is rationally equivalent to $\frac{1}{4}d_{2,2,1\bullet} + \frac{3}{4}d_{3,1,1\bullet}$. $\square$

Note that

$\rho_*\phi_1^*[\overline{\mathcal{B}}_{2,1,0}] = -3\rho_*\phi_1^*$  .

Corollary 3.4.6. We have

$[\overline{\mathcal{B}}_{2,1,0}] \in \Big\{ -3$  $+ c \;\Big|\; c \in \ker \phi^* \Big\} =$

$\Big\{ a$  $+ 14a$  $- 2a$  $+ (36a-3)$  $+ b$  $\Big\}$.

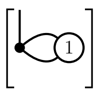
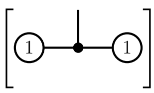
where ϕ is the matrix of Proposition 3.4.4

Proposition 3.4.7. In the statement of Corollary 3.4.6 we have $a = 1/4$ and $b = 24$.

Proof. The coefficients a and b can be determined by taking the pushforward of $\overline{\mathcal{B}}_{2,1,0}$ under the forgetful map $\pi\colon \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_2$. We have $\pi_*[\overline{\mathcal{B}}_{2,1,0}] = 4[\overline{\mathcal{B}}_2]$ since every bielliptic curve of genus 2 has two distinct bielliptic involutions (permuted by the hyperelliptic involution) and there are two choices of a ramification point. It is known (see for example [FP15]) that

$$[\overline{\mathcal{B}}_2] = \frac{3}{2}\delta_0 + 6\delta_1.$$

It is easy to see that the pushforward under π of the first three basis elements of $A^2(\overline{\mathcal{M}}_{2,1})$ (in the basis of 3.4) is zero (since the image of the corresponding loci drops dimension). The pushforward of

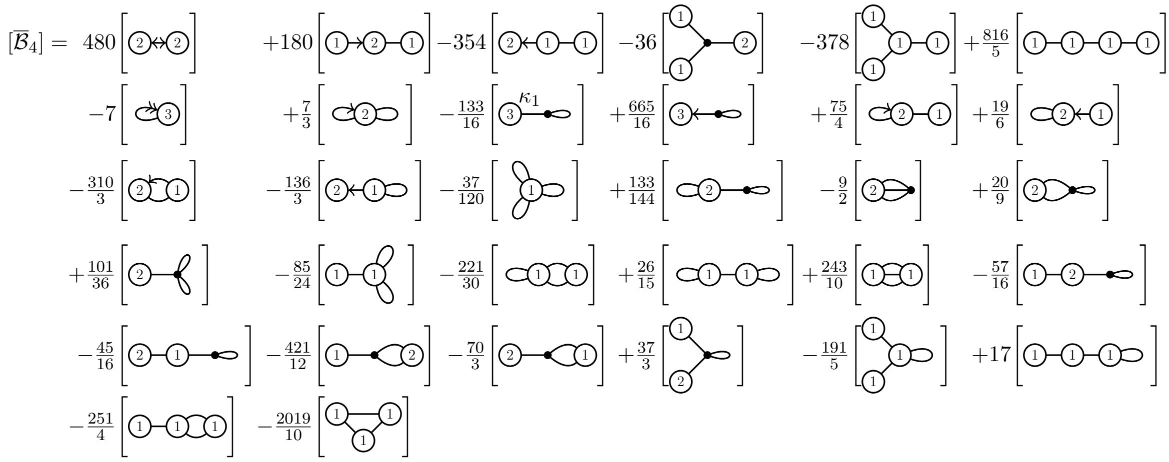
 and 

is δ_0 and δ_1 respectively. From this we read off that $a = 1/4$ and $b = 24$. $\square$

The Class of $\overline{\mathcal{B}}_4$

We now turn to the most difficult class we will explicitly compute. The dimension of $H^6(\overline{\mathcal{M}}_4)$ is 32 by [BT07, Theorem 1]. By [Yan08, Page 11] the dimension of $RH^6(\overline{\mathcal{M}}_4)$ is also 32, hence the cohomology group is completely tautological. A basis for $H^6(\overline{\mathcal{M}}_4)$ in terms of decorated stratum classes is given by taking the last 32 generators of the relation given in [Yan08, Proposition 2].

Theorem 3.4.8. In $H^6(\overline{\mathcal{M}}_4)$ we have



Remark 3.4.9. We only know that the equality of Theorem 3.4.8 holds in cohomology. Because the entire cohomology ring is tautological we know that the cohomology class $[\overline{\mathcal{B}}_4]$ is tautological. However the author is unaware of any argument that the corresponding Chow class is tautological.

By definition we have $\dim R^3(\overline{\mathcal{M}}_4) \geq \dim RH^6(\overline{\mathcal{M}}_4)$. From analyzing the relations and generators of $R^3(\overline{\mathcal{M}}_4)$ it follows that $\dim R^3(\overline{\mathcal{M}}_4) \leq 32$. We therefore deduce that $R^3(\overline{\mathcal{M}}_4) = RH^6(\overline{\mathcal{M}}_4)$. In particular $A^3(\overline{\mathcal{M}}_4)$ splits as $R^3(\overline{\mathcal{M}}_4) \oplus \ker \mathrm{cyc}$ where $\mathrm{cyc}\colon A^3(\overline{\mathcal{M}}_4) \to H^6(\overline{\mathcal{M}}_4)$ is the cycle map (see Remark 1.3.10). The equality of Theorem 3.4.8 therefore also holds in $A^3(\overline{\mathcal{M}}_4)/\ker(\mathrm{cyc})$.

To prove Theorem 3.4.8 we will first do a pullback to the hyperelliptic locus. We will then do a number of pullbacks along boundary strata as in Section 3.3. Together these will completely determine the class $[\overline{\mathcal{B}}_4]$.

Notation 3.4.10. We adopt again the notation of 2.2.28. A set of generators for $H^6(\overline{\mathcal{M}}_{0,10})$ is given by $d_{i,j,k,l}$ with $i+j+k+l=10$, $i,l \geq 2$, $j,k \geq 1$ and $e_{i,j,k,l}$ with $i+j+k+l=10$, $j,k,l \geq 2$. There are 30 generators of this kind. A basis for $H^6(\overline{\mathcal{M}}_{0,10})^{\mathfrak{S}_{10}}$ can be computed using the fundamental relation $(12|34) = (13|24) = (14|23)$ in $A^1(\overline{\mathcal{M}}_{0,4})$ (see Example 1.3.7). By pulling this relation back we get the following relations among the elements of $H^6(\overline{\mathcal{M}}_{0,10})$:

$$\begin{aligned} 4e_{0,4,2,2} &= d_{4,1,1,2} & & \\ 2e_{0,5,3,2} &= d_{5,1,1,3} & 2e_{0,4,4,2} &= d_{4,1,1,4} \\ d_{5,2,1,2} &= d_{5,1,2,2} & d_{4,2,1,3} &= d_{4,1,2,3} \\ 2e_{0,5,3,2} + 3e_{1,5,2,2} &= 12d_{5,2,1,2} & e_{0,4,3,3} + 6e_{1,4,3,2} &= 12d_{4,2,1,3} \end{aligned}$$

$$\begin{aligned} 6e_{0,4,4,2} + 16e_{1,4,3,2} + 6e_{2,4,2,2} &= 8d_{4,3,1,2} + 24d_{4,2,2,2} + 8d_{4,1,3,2} \\ 3e_{0,4,3,3} + 8e_{1,3,3,3} + 18e_{2,3,3,2} &= 12d_{3,3,1,3} + 36d_{3,2,2,3} \end{aligned}$$

Note that these relations are independent. It is known that $\dim H^6(\overline{\mathcal{M}}_{0,10}) = 21$ (a procedure for computing $\dim H^{2k}(\overline{\mathcal{M}}_{0,n})$ was given in [Get95], this has been implemented in an unpublished Maple program by Jonas Bergstrom). Therefore these relations are all relations among the above generators. By elimination we now fix the following basis for $H^6(\overline{\mathcal{M}}_{0,10})$:

$$\begin{matrix} d_{6,1,1,2} & d_{5,2,1,2} & d_{5,1,2,2} & d_{5,1,1,3} & d_{4,3,1,2} & d_{4,2,2,2} & d_{4,2,1,3} \\ d_{4,1,3,2} & d_{4,1,2,3} & d_{4,1,1,4} & d_{3,4,1,2} & d_{3,3,2,2} & d_{3,3,1,3} & d_{3,2,3,2} \\ d_{3,2,2,3} & d_{3,1,4,2} & d_{2,5,1,2} & d_{2,4,2,2} & d_{2,3,3,2} & e_{1,3,3,3} & e_{0,4,3,3} \end{matrix}$$

Proposition 3.4.11. The pull-push $\rho_* \circ \phi_0^*\colon H^6(\overline{\mathcal{M}}_4) \to H^6(\overline{\mathcal{M}}_{0,10})$ is given by the matrix (in terms of the bases of decorated stratum classes we fixed at the start of this subsection and the invariant classes given in Notation 3.4.10):

$$\left(\begin{array}{cccccccccccccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \frac{13}{21} & 6 & \frac{13}{14} & \frac{5}{14} & 0 & 0 & 0 & 0 & -240 & 4 & 6 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{21} & 0 & \frac{61}{42} & \frac{5}{21} & 0 & 0 & 0 & 0 & -320 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{21} & \frac{8}{3} & 0 & 0 & 0 & 0 & -\frac{1}{9} & 0 & -320 & 0 & 0 & -\frac{8}{3} & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} & 0 & 960 & 0 & 0 & 0 & -6 & -20 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{71}{210} & 7 & \frac{11}{7} & \frac{1}{7} & 0 & 0 & -\frac{1}{60} & 0 & -144 & 4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{26}{105} & \frac{124}{15} & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & 0 & 0 & 0 & -\frac{8}{5} & 0 & 0 & -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{15} & 2 & 0 & 0 & \frac{11}{30} & \frac{1}{3} & -\frac{4}{15} & 0 & 864 & 0 & 0 & 0 & 0 & -8 & -2 & 0 & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 \\
\frac{3}{400} & 0 & 0 & 0 & 0 & 0 & \frac{11}{70} & 5 & 0 & 0 & 0 & 0 & -\frac{1}{20} & \frac{7}{20} & -144 & 0 & 0 & 0 & 0 & 0 & -3 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{9}{400} & 0 & \frac{3}{80} & 0 & 0 & 0 & \frac{2}{15} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{6} & -\frac{2}{15} & \frac{3}{20} & 864 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{3} & 0 \\
\frac{9}{200} & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & -\frac{2}{5} & \frac{3}{5} & 576 & 0 & 0 & 0 & 0 & 0 & -4 & 8 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{35} & 0 & \frac{9}{7} & \frac{1}{14} & \frac{4}{5} & \frac{1}{10} & 0 & 0 & -192 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{70} & \frac{4}{5} & 0 & 0 & \frac{41}{60} & \frac{1}{5} & -\frac{1}{30} & 0 & -288 & 0 & 0 & -\frac{4}{5} & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{12} & 0 \\
0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{20} & 0 & -\frac{1}{10} & 0 & 864 & 0 & 0 & 0 & 0 & -36 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{10} & 0 & 0 & \frac{5}{2} & 0 & -\frac{1}{4} & 0 \\
\frac{3}{800} & 0 & \frac{1}{160} & 0 & 0 & 0 & \frac{3}{35} & 0 & 0 & 0 & \frac{2}{5} & \frac{3}{10} & 0 & \frac{3}{20} & -288 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
\frac{9}{800} & \frac{1}{16} & \frac{3}{80} & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1152 & 0 & 0 & 0 & 0 & -48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{70} & 2 & 0 & 0 & \frac{7}{30} & \frac{2}{5} & -\frac{1}{30} & 0 & -192 & 0 & 0 & 0 & 0 & 16 & -2 & 0 & 0 & 0 & 0 & -\frac{1}{30} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{14} & 8 & \frac{25}{42} & \frac{1}{42} & 0 & 0 & 0 & 0 & 160 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{14} & \frac{134}{15} & 0 & 0 & 0 & 0 & -\frac{1}{90} & 0 & 448 & 0 & 0 & -\frac{4}{15} & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{800} & 0 & 0 & 0 & 0 & 0 & \frac{1}{14} & 8 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10} & 288 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5184 & 0 & 0 & 0 & 0 & 216 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8640 & 0 & 0 & 0 & 0 & 240 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0
\end{array}\right)$$

Proof. This is an application of Theorem 2.2.26. □

Proposition 3.4.12. The inverse image of $\overline{\mathcal{B}}_4$ under the map $\phi_0\colon \mathcal{Adm}(4,0) \to \overline{\mathcal{M}}_4$ is given by two strata:

1. the locus $\mathcal{A}$ of admissible covers $S \to T$ where S is a curve with two irreducible components S_1 and S_2 of genus 1 and 2 respectively and two nodes between them; and where S_2 admits a bielliptic involution switching the nodes,

2. the locus $\mathcal{B}$ of admissible covers $S \to T$ where S is a curve with two irreducible components S_1 and S_2 of genus 1 and 2 and two nodes between them; and where S_1 admits a bielliptic involution switching the nodes.

Proof. This follows from a combinatorial exhaustion as in the proof of Proposition 3.4.5. □

Notation 3.4.13. We will denote by

$$\overline{\mathcal{H}}^s_{0,1,2m} := \bigcup_{\sigma \in \mathfrak{S}_{2m}} \sigma(\overline{\mathcal{H}}_{0,1,2m}) \subset \overline{\mathcal{M}}_{0,1+2m}$$

the image under the action of the symmetric group $\mathfrak{S}_{2m}$ acting on the $2m$-markings permuted by the hyperellipitc involution.

Proposition 3.4.14. Let $\xi\colon \overline{\mathcal{M}}_{0,7} \times \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,10}$ be the gluing morphism gluing the curves together in the last points. Let $\rho\colon \mathcal{Adm}(4,0) \to \overline{\mathcal{M}}_{0,10}$ be the target map. We have (with the notation of Proposition 3.4.12)

$$\begin{aligned}
\rho(\mathcal{A}) &= \xi(\overline{\mathcal{H}}^s_{0,1,7} \times \overline{\mathcal{M}}_{0,5}) \\
\rho(\mathcal{B}) &= \xi(\overline{\mathcal{M}}_{0,7} \times \overline{\mathcal{H}}^s_{0,1,4}).
\end{aligned}$$

Proof. We adopt the notation of Proposition 3.4.12 and set T_1 and T_2 to be the images of S_1 and S_2 under the hyperelliptic map. The loci $\rho(\mathcal{A})$ and $\rho(\mathcal{B})$ are obviously contained in

$\xi(\overline{\mathcal{M}}_{0,7} \times \overline{\mathcal{M}}_{0,5})$. In the case of $\mathcal{A}$ there are no conditions on the curve T_1 and we must find the locus of curves T_2 in $\overline{\mathcal{M}}_{0,7}$ lifting to curves S_2 admitting a bielliptic involution (which switches the marked points exchanged by the hyperelliptic involution). Similarly for $\mathcal{B}$ there are no conditions on T_2 and we must find the locus of curves T_2 in $\overline{\mathcal{M}}_{0,5}$ lifting to curves S_2 admitting a bielliptic involution.

The bielliptic involution commutes with the hyperelliptic involution in any genus therefore the set of ramification points of the hyperelliptic involution is fixed by the bielliptic involution. However bielliptic involutions do not have any fixed points on curves of genus 1, while on a curve of genus 2 the bielliptic involution does not fix any of the ramification points of the hyperelliptic involution (see the proof of 3.4.5).

In the case of $\mathcal{A}$ passing the bielliptic involution through the hyperelliptic map $S_2 \to T_2$ we therefore get the locus of curves admitting an involution permuting the first 6 points and fixing the last. In the case of $\mathcal{B}$ we get the locus of curves inside $\overline{\mathcal{M}}_{0,5}$ admitting an involution permuting the first 4 points and fixing the last point. $\square$

3.4.15. Consider the fiber square

$$\begin{array}{ccc} \mathcal{A} \cup \mathcal{B} & \xrightarrow{g} & \overline{\mathcal{B}}_4 \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow \\ \mathcal{A}dm(4,0) & \xrightarrow{\phi_0} & \overline{\mathcal{M}}_4 \end{array}.$$

We have $\operatorname{codim}_{\overline{\mathcal{M}}_4} \overline{\mathcal{B}}_4 = 3$. Since $\operatorname{codim}_{\overline{\mathcal{M}}_{0,10}} \xi(\overline{\mathcal{H}}^s_{0,1,6} \times \overline{\mathcal{M}}_{0,5}) = 3$ the excess bundle $E = i^* N_{\phi_0}/N_g$ is trivial on $\mathcal{A}$.

We have $\operatorname{codim}_{\overline{\mathcal{M}}_{0,10}} \xi(\overline{\mathcal{M}}_{0,5} \times \overline{\mathcal{H}}^s_{0,1,4}) = 2$. Let $\tilde{\phi}_0\colon \mathcal{A}dm(2,0)_2 \to \overline{\mathcal{M}}_{2,2}$ be the admissible map and let $p\colon \mathcal{B} \to \mathcal{A}dm(2,1)$ be the projection onto the second factor. The excess bundle restricted to $\mathcal{B}$ is given by

$$p^* N_{\tilde{\phi}_0}.$$

From the excess intersection formula (see Proposition 1.2.13) it now follows that:

Proposition 3.4.16. Let $\tilde{\rho}\colon \mathcal{A}dm(2,0)_2 \to \overline{\mathcal{M}}_{0,7}$ be the target map. With the notation of Proposition 3.4.14 and of Paragraph 3.4.15, we have

$$\rho_* \phi_0^*([\overline{\mathcal{B}}_4]) = \xi_*([\overline{\mathcal{H}}^s_{0,1,6}] \otimes [\overline{\mathcal{M}}_{4,1}] + (\rho_* c_1(N_{\tilde{\phi}_0}) \cap [\overline{\mathcal{M}}_{0,7}]) \otimes [\overline{\mathcal{H}}^s_{0,1,4}]).$$

Lemma 3.4.17. We have

$$[\overline{\mathcal{H}}^s_{0,1,4}] = \frac{1}{2} d_{2,2\bullet} + \frac{3}{2} d_{3,1\bullet}.$$

Proof. The argument is well known. We will show that $[\overline{\mathcal{H}}_{0,1,4}] = (12|34\bullet) + (123|4\bullet) + (124|3\bullet)$ (where we use the notation from Example 1.3.7 and $\bullet$ is the marked point fixed by the involution) the result then follows from the definition of $[\overline{\mathcal{H}}^s_{0,1,4}]$, $d_{2,2\bullet}$ and $d_{3,1\bullet}$. We can identify $\overline{\mathcal{M}}_{0,5}$ with the blowup of $\mathbb{P}^2$ in the following way: Recall that conics in $\mathbb{P}^2$ are rational curves and any 5 points or any 4 points together with a tangent direction at a point in $\mathbb{P}^2$ define a unique conic. Choose four points P_1, P_2, P_3, P_4 in general position in $\mathbb{P}^2$ and let a fifth point $\bullet$ vary across $\mathbb{P}^2 \backslash \{P_1, ..., P_4\}$. We have a map $\mathbb{P}^2 \backslash \{P_1, P_2, P_3, P_4\} \hookrightarrow \overline{\mathcal{M}}_{0,5}$ which send $\bullet$ to the conic determined by $\{P_1, ..., P_4, \bullet\}$. The open part $\mathcal{M}_{0,5}$ is contained in the image of this map. To get the closure we note that when $\bullet$ approaches one of the points P_1, P_2, P_3, P_4 the tangent direction at which $\bullet$ approaches this point determines a unique conic and therefore an element of $\overline{\mathcal{M}}_{0,5}$. In this way we have identified $\overline{\mathcal{M}}_{0,5}$ with the blowup $\tilde{\mathbb{P}}^2$ of $\mathbb{P}^2$ in the four points P_1, P_2, P_3, P_4.

Let $\pi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blowdown map. The invariant locus of the involution $(12)(34)(\bullet)$ is the proper transform of a line H in $\mathbb{P}^2$ which does not cross any of the points P_1, P_2, P_3, P_4. Consider $(12|34\bullet) \subset \overline{\mathcal{M}}_{0,5} \simeq \tilde{\mathbb{P}}^2$; its image under π is the line l in $\mathbb{P}^2$ such that if we place $\bullet$ on this line the resulting conic through $P_1, P_2, P_3, P_4, \bullet$ consists of a line through P_1 and P_2 and a line through $P_3, P_4, \bullet$. The strict transform of l in $\tilde{\mathbb{P}}^2$ therefore contains the exceptional divisors E_3 and E_4 where $\bullet$ approaches P_3 and P_4 respectively. But these exceptional divisors are exactly $(124|3\bullet)$ and $(123|4\bullet)$. Now note that the strict transform of a general line in $\mathbb{P}^2$ is rationally equivalent to a a line going through P_3 and P_4 so $(12|34\bullet) + (123|4\bullet) + (124|3\bullet)$. $\square$

Lemma 3.4.18. We have

$$\begin{aligned}[\overline{\mathcal{H}}^s_{0,1,6}] =& 10d_{4,1,1\bullet} + 24d_{3,2,1\bullet} + 16d_{3,1\bullet,2} + 6d_{2,2,2\bullet} \\ &+ 4d_{3,1,2\bullet} + (4/3)d_{3,\bullet,3} + 8d_{2,3,1\bullet} + 3d_{2,2\bullet,2} + ad_{4,\bullet,2} + bd_{3,\bullet,3}\end{aligned}$$

for some scalars $a, b \in \mathbb{Q}$.

Proof. We follow the proof of [FP15, Lemma 4]. Let $\pi_{12}, \pi_{34} \colon \overline{\mathcal{M}}_{0,7} \to \overline{\mathcal{M}}_{0,5}$ be the forgetful maps forgetting the points $1, 2$ and $3, 4$ respectively. We have an equality

$$\mathcal{H}_{0,1,6} = \pi_{12}^{-1}(\mathcal{H}_{0,1,4}) \cap \pi_{34}^{-1}(\mathcal{H}_{0,1,4}).$$

Indeed curves in the set on the right admit two involutions σ, τ fixing the point $\bullet$ and sending 5 to 6. The involution σ also sends 1 to 2 and the involution τ sends 3 to 4. Since involutions on $\mathbb{P}^1$ fixing one point and permuting a pair of points are unique we have $\sigma = \tau$ and they act on the set of marked points as $(12)(34)(56)(\bullet)$.

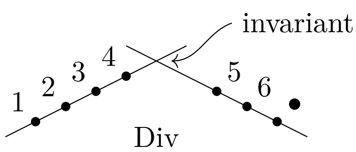
This shows that $\overline{\mathcal{H}}_{0,1,6}$ is an irreducible component of

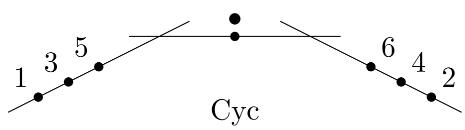
$$\pi_{12}^{-1}(\overline{\mathcal{H}}_{0,1,4}) \cap \pi_{34}^{-1}(\overline{\mathcal{H}}_{0,1,4}).$$

There are 2 more irreducible components of $\pi_{12}^{-1}(\overline{\mathcal{H}}_{0,1,4}) \cap \pi_{34}^{-1}(\overline{\mathcal{H}}_{0,1,4})$:

1. The locus Div whose curves consist of curves inside $(1234|56\bullet)$ admitting an involution on the component containing 5, 6 and $\bullet$ which switches 5 and 6 and fixes the node and $\bullet$. There always exist an involution on the left side which switches 1 and 2 and fixes the node, or switches 3 and 4 and fixes the node so this locus is contained in $\pi_{12}^{-1}(\overline{\mathcal{H}}_{0,1,4}) \cap \pi_{34}^{-1}(\overline{\mathcal{H}}_{0,1,4})$. But these involutions do not neccesarily coincide so this locus is not contained in $\overline{\mathcal{H}}_{0,1,6}$.

2. The locus Cyc consisting of curves $(123|\bullet|456)$. This locus lies generically in $\pi_{12}^{-1}(\overline{\mathcal{H}}_{0,1,4})$ and $\pi_{34}^{-1}(\overline{\mathcal{H}}_{0,1,4})$ since there is always an involution on the middle curve which fixes the point $\bullet$ and switches the nodes and there is always a map between two copies of $\mathbb{P}^1$ containing three specified points each which sends specified points to specified points. Since there is not generically a map between 2 copies of $\mathbb{P}^1$ with 4 marked points which sends the marked points to the marked points this locus is not contained in $\overline{\mathcal{H}}_{0,1,4}$.





It is straightforward to check that none of the other codimension 2-boundary components are generically contained in $\pi_{12}^{-1}(\overline{\mathcal{H}}_{0,1,4}) \cap \pi_{34}^{-1}(\overline{\mathcal{H}}_{0,1,4})$ and none of the boundary component have a set theoretic intersection with $\pi_{12}^{-1}(\overline{\mathcal{H}}_{0,1,4}) \cap \pi_{34}^{-1}(\overline{\mathcal{H}}_{0,1,4})$ of codimension at most 1.

We have thus proven that

$$[\overline{\mathcal{H}}_{0,1,6}] = \pi_{12}^{-1}(\overline{\mathcal{H}}_{0,1,4}) \cdot \pi_{34}^{-1}(\overline{\mathcal{H}}_{0,1,4}) - a[\mathrm{Div}] - b[\mathrm{Cyc}].$$

The intersection $\pi_{12}^{-1}(\overline{\mathcal{H}}_{0,1,4}) \cdot \pi_{34}^{-1}(\overline{\mathcal{H}}_{0,1,4})$ can be determined using 2.1.30. The cycle [Cyc] is defined by the dual graph of the curves in Cyc. For [Div] we need to calculate the locus $[\overline{\mathcal{H}}_{0,2,2}]$. From Proposition 3.2.8 and from the calculation of $[\overline{\mathcal{H}}_{0,1,4}]$ it follows immediatly that $[\overline{\mathcal{H}}_{0,2,2}] = (12| \star \bullet)$, where $\star$ and $\bullet$ are the points fixed by the involution, and therefore $[\mathrm{Div}] = (1234| \bullet |56)$. □

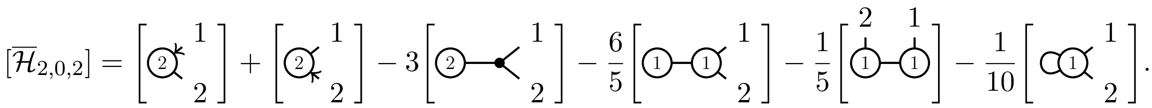
Lemma 3.4.19. Let $\tilde{\phi}_0\colon \mathcal{A}dm(2,0)_2 \to \overline{\mathcal{M}}_{2,2}$ be the source map and $\rho\colon \mathcal{A}dm(2,0)_2 \to \overline{\mathcal{M}}_{0,7}$. We have

$$\rho_* c_1(N_{\tilde{\phi}_0}) \cap [\overline{\mathcal{M}}_{0,7}] = -\frac{1}{6} d_{5,1\bullet} - \frac{1}{15} d_{4\bullet,2} + \frac{3}{5} d_{4,2\bullet} - \frac{1}{5} d_{3,3\bullet}. \tag{3.8}$$

Proof. We have

$$\rho_* c_1(N_{\tilde{\phi}_0}) \cap [\overline{\mathcal{M}}_{0,7}] = \rho_* \tilde{\phi}_0^*([\overline{\mathcal{H}}_{2,0,2}])$$

The class $[\overline{\mathcal{H}}_{2,0,2}] \in A^1(\overline{\mathcal{M}}_{2,2})$ was originally computed in [BP00, Lemma 6]. After a change of basis we have:



(3.9)

The composition $\rho_* \tilde{\phi}_0^*\colon A^1(\overline{\mathcal{M}}_{2,2}) \to A^1(\overline{\mathcal{M}}_{0,7})$ is given by Theorem 2.2.26. Explicitly we have (using the same bases as those of Equation (3.9) and Equation (3.8))

$$\rho_* \circ \tilde{\phi}_0^* = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{15} & \frac{1}{15} & 0 & 0 & 0 & 2 \\ \frac{2}{5} & \frac{2}{5} & 0 & 0 & 0 & 2 \\ \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

□

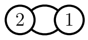
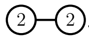
3.4.20. To finish the computation of $[\overline{\mathcal{B}}_4]$ in terms of decorated stratum classes we will do a number of pullbacks to boundary strata (in the same way as we did in Section 3.3). The Castelnuovo-Severi inequality says that if S is a smooth curve of genus g with two distinct covering morphisms f_1, f_2 to curves T_1 and T_2 of genus h_1 and h_2 and if $\deg f_1 = d_1$ and $\deg f_2 = d_2$, then the following inequality holds:

$$g \leq d_1 h_1 + d_2 h_2 + (d_1 - 1)(d_2 - 1)$$

It follows that bielliptic involutions on (smooth) curves of genus 4 are unique, therefore

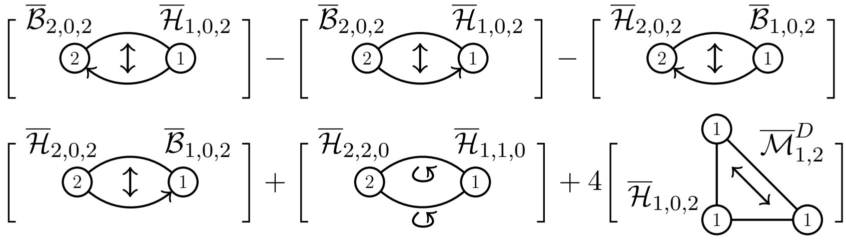
$$\deg \mathcal{A}dm(g,1)/\overline{\mathcal{B}}_4 = |\mathfrak{S}_6| = 6!$$

Let

$A =$  $\qquad\qquad B =$ .

The pullback maps $\xi_A^*\colon H^6(\overline{\mathcal{M}}_4) \to H^6(\overline{\mathcal{M}}_A)$ and $\xi_B^*\colon H^6(\overline{\mathcal{M}}_4) \to H^6(\overline{\mathcal{M}}_B)$ in terms of bases of decorated stratum classes can be computed by Theorem 2.1.30.

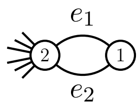
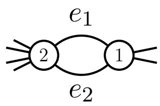
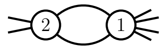
Lemma 3.4.21. The pullback of $[\overline{\mathcal{B}}_4]$ along ξ_A is given by:

$\xi_A^*([\overline{\mathcal{B}}_4]) = -$ 

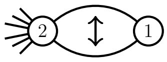
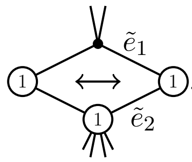
$$
\begin{aligned}
&= -\psi_1[\overline{\mathcal{B}}_{2,0,2}] \otimes [\overline{\mathcal{H}}_{1,0,2}] - [\overline{\mathcal{B}}_{2,0,2}] \otimes \psi_1[\overline{\mathcal{H}}_{1,0,2}] - \psi_1[\overline{\mathcal{H}}_{2,0,2}] \otimes [\overline{\mathcal{B}}_{1,0,2}] \\
&\quad - [\overline{\mathcal{H}}_{2,0,2}] \otimes \psi_1[\overline{\mathcal{B}}_{1,0,2}] + [\overline{\mathcal{H}}_{2,2,0}] \otimes [\overline{\mathcal{H}}_{1,0,2}] + 2[\overline{\mathcal{H}}_{1,0,2} \times \overline{\mathcal{M}}_{1,1}^D].
\end{aligned}
$$

Where $\overline{\mathcal{M}}_{1,1}^D$ is as in notation 2.2.13.

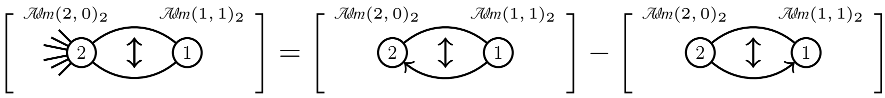
Proof. This is an application of Theorem 2.2.21. The set $\mathfrak{A}$ consists of all possible distributions of legs on A. We have the following possibilities:

$B_1 =$  $\qquad B_2 =$  $\qquad B_3 =$ .

The following admissible bielliptic pairs admit a B_1-structure:

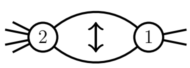
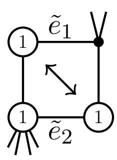
$(\Gamma_{1,1}, \tau_{1,1}) =$  $\qquad (\Gamma_{1,2}, \tau_{1,2}) =$ .

There is one isomorphism class of B_1-structures on $(\Gamma_{1,1}, \tau_{1,1})$. The top chern class of the excess bundle $E_{(\Gamma_{1,1},\tau_{1,1})}$ is given by a ψ class on each side. In other words

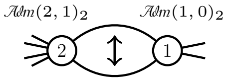
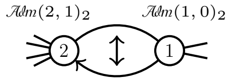
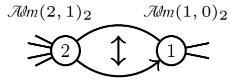
$c_1(E_{\Gamma_{1,1},\tau_{1,1}}) \cap$ 

There are 2 isomorphism classes of B_1-structures on $(\Gamma_{1,2}, \tau_{1,2})$. Indeed a B_1-structure $f = (\alpha, \beta, \gamma)$ on $(\Gamma_{1,2}, \tau_{1,2})$ can be given either by sending e_1 to $\tilde{e}_1$ or to $\tilde{e}_2$ under β. It is easy to see they are not isomorphic.

The admissible bielliptic pairs with a B_2-structure are given by

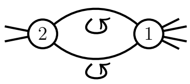
$(\Gamma_{2,1}, \tau_{1,1}) =$  $\qquad (\Gamma_{2,2}, \tau_{2,2}) =$ 

Again there is only one isomorphism class of B_2-structures on $(\Gamma_{2,1}, \tau_{2,1})$ and the excess bundle is given by

$E_{(\Gamma_{2,1},\tau_{1,1})} \cap$ [] $= -$ [] $-$ []

There are 2 isomorphsim classes of B_2-structures on $(\Gamma_{2,2}, \tau_{2,2})$. We can send the edge e_1 to either $\tilde{e}_1$ or $\tilde{e}_2$ and this completely determines the B_2-structure.

There is only one admissible pair $(\Gamma_{3,1}, \tau_{3,1})$ with a B_3-structure. It is given by

$(\Gamma_3, \tau_3) =$ 

There is only one isomorphism class of B_3-structures and the top Chern class of the excess bundle is trivial.

We have determined the sum of the classes in the expression of Theorem 2.2.21. Pushing all of these classes forward through the morphisms $\pi_B \circ \phi_f$ we obtain the desired expression. $\square$

Remark 3.4.22. We computed the class $[\overline{\mathcal{B}}_{2,0,2}]$ in terms of decorated stratum classes in Proposition 3.2.10. We have $[\overline{\mathcal{H}}_{1,0,2}] = [\overline{\mathcal{M}}_{1,0,2}]$. We computed the class $[\overline{\mathcal{H}}_{2,0,2}]$ in 3.4.19 and the class $[\overline{\mathcal{B}}_{1,0,2}]$ in Remark 3.4.2. The class $[\overline{\mathcal{H}}_{2,2,0}]$ is given as $[\overline{\mathcal{DR}}_2(2)]$ in [Tar15, Theorem 0.1]. The class $[\overline{\mathcal{H}}_{1,0,2}]$ is given as $[\bar{A}_2]$ in [Pag13, Theorem 3.33]. The class of the diagonal $[\overline{\mathcal{M}}^D_{1,2}]$ can easily be determined using Proposition 1.3.11. We have thus completely determined the class $\xi_A^*([\overline{\mathcal{B}}_4])$ in terms of decorated stratum classes.

Lemma 3.4.23. We have

$$\xi_B^*([\overline{\mathcal{B}}_4]) = [\overline{\mathcal{B}}_{2,1,0} \times \overline{\mathcal{H}}_{2,1,0}] + [\overline{\mathcal{H}}_{2,1,0} \times \overline{\mathcal{B}}_{2,1,0}] \in H^6(\overline{\mathcal{M}}_{2,1} \times \overline{\mathcal{M}}_{2,1}).$$

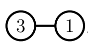
Proof. This is Example 2.2.22. $\square$

Proof of Theorem 3.4.8. We now have three linear maps ξ_A^*, ξ_B^* and $\rho_*\phi_0^*$ which we can determine in terms of a basis of decorated stratum classes for $H^6(\overline{\mathcal{M}}_4)$. It turns out that

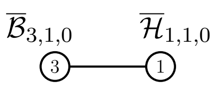
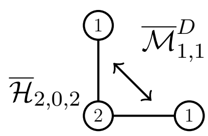
$$\ker(\xi_A^*) \cap \ker(\xi_B^*) \cap \ker(\rho_*\phi_0^*) = \{0\}.$$

We have also determined the the pullbacks $\xi_A^*([\overline{\mathcal{B}}_4])$, $\xi_B^*([\overline{\mathcal{B}}_4])$ and $\rho_*\phi_0^*([\overline{\mathcal{B}}_4])$ in terms of decorated stratum classes. Together this completely determines the class $[\overline{\mathcal{B}}_4]$. $\square$

Remark 3.4.24. We can also consider the graph

$C =$ .

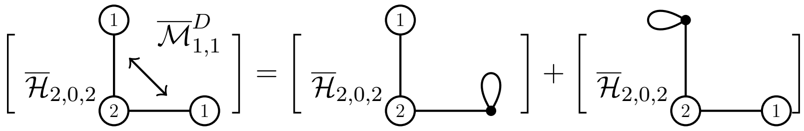
In $H^6(\overline{\mathcal{M}}_{3,1} \times \overline{\mathcal{M}}_{1,1})$ we have

$\xi_C^*([\overline{\mathcal{B}}_4]) =$ [] $+ 2$ []

$$=[\overline{\mathcal{B}}_{3,1,0} \times \overline{\mathcal{H}}_{1,1,0}] + 2[\overline{\mathcal{H}}_{2,0,2} \times \overline{\mathcal{M}}^D_{1,1}]$$

The class $\overline{\mathcal{B}}_{3,1,0}$ has not been computed before. We could determine $\overline{\mathcal{B}}_{3,1,0}$ using Theorem 3.4.8 and Proposition 3.2.2. Conversely we could also use Theorem 2.2.21 without using the pullback to the hyperelliptic locus by first computing $[\overline{\mathcal{B}}_{3,1,0}]$ using repeated pullbacks as in Section 3.3 and/or pushforward along forgetful morphisms (such as $\pi\colon \overline{\mathcal{M}}_{3,1} \to \overline{\mathcal{M}}_3$).

We can still obtain information by the pullback ξ_C^*. In particular $[\overline{\mathcal{B}}_{3,1,0}]\otimes[\overline{\mathcal{H}}_{1,1,0}] \in H^6(\overline{\mathcal{M}}_{3,1})\otimes H^0(\overline{\mathcal{M}}_{1,1})$, while



Under the identification $H^6(\overline{\mathcal{M}}_{3,1} \times \overline{\mathcal{M}}_{1,1}) = \bigoplus_i H^{6-i}(\overline{\mathcal{M}}_{3,1}) \otimes H^i(\overline{\mathcal{M}}_{1,1})$ we see that the second class on the right hand side is an element of $H^6(\overline{\mathcal{M}}_{3,1})\otimes H^0(\overline{\mathcal{M}}_{1,1})$ while the first is an element of $H^4(\overline{\mathcal{M}}_{3,1}) \otimes H^2(\overline{\mathcal{M}}_{1,1})$. Let $p_i\colon H^6(\overline{\mathcal{M}}_{3,1} \times \overline{\mathcal{M}}_{1,1}) \to H^{6-i}(\overline{\mathcal{M}}_{3,1}) \otimes H^i(\overline{\mathcal{M}}_{1,1})$ be the projection. We can thus compute the composition $p_{2i*}\xi_C^*([\overline{\mathcal{B}}_4])$. This is not enough to determine the class $[\overline{\mathcal{B}}_4]$ without pulling back to the hyperelliptic locus but it does provide $\mathrm{rank}(p_{2*}\xi_C^*) = 15$ an extra check on the calculation.

– IV –

Nontautological Bielliptic Classes

In this chapter we will prove that for sufficiently high $g, n, 2m$ the class of the locus of bielliptic curves is nontautological. This chapter has appeared in Pacific Journal of Mathematics as a separate paper [vZ18].

4.1. In Chapter 3 we have computed classes of admissible covers in terms of decorated stratum classes. This is only possible if the class is tautological. In the cases where we made these computations we knew these classes were tautological, however this is not always the case. Indeed Deligne proved that $H^{11}(\overline{\mathcal{M}}_{1,11}) \neq 0$, thus providing a first example of the existence of nontautological classes. In fact it is known that $H^{\bullet}(\overline{\mathcal{M}}_{0,n}) = RH^{\bullet}(\overline{\mathcal{M}}_{0,n})$ (see [Kee92]) and that $H^{2\bullet}(\overline{\mathcal{M}}_{1,n}) = RH^{2\bullet}(\overline{\mathcal{M}}_{1,n})$ (see [Pet14, Corollary 1.2]).

Examples of geometrically defined loci which can be proven to be nontautological are still relatively scarce. In [GP03] Graber and Pandharipande hunt for algebraic classes in $H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ and in $H^{2\bullet}(\mathcal{M}_{g,n})$ which are nontautological. In particular, they show that the classes of the loci $\overline{\mathcal{B}}_{g,n,2m}$ and $\mathcal{B}_{g,n,2m} := \overline{\mathcal{B}}_{g,n,2m}|_{\mathcal{M}_{g,n+2m}}$ of stable respectively smooth bielliptic curves are nontautological when $g = 2$, $n = 0$ and $2m = 20$ (i.e. $[\overline{\mathcal{B}}_{2,0,20}] \notin RH^{\bullet}(\overline{\mathcal{M}}_{2,20})$ and $[B_{2,0,20}] \notin RH^{\bullet}(\mathcal{M}_{2,20})$). They also show that for sufficiently high odd genus h the class of $\phi_0(\mathcal{A}dm(2h, h))$ is nontautological in $\overline{\mathcal{M}}_{2h}$. Their result relies on the existence of odd cohomology in $H^{\bullet}(\overline{\mathcal{M}}_{h,1})$, which was proven in [Pik95] for all $h \geq 8069$. See [FP13] for a recent survey of different methods of detecting nontautological classes.

In [PT14, Pet16] Petersen and Tommasi proved that $H^{2\bullet}(\overline{\mathcal{M}}_{2,n})$ is tautological for all $n < 20$ and that $H^{2\bullet}(\overline{\mathcal{M}}_{2,20})$ is additively generated by tautological classes, by the class $[\overline{\mathcal{B}}_{2,0,20}]$, and by its conjugates under the action of the symmetric group on 20 elements. In this sense the result of Graber and Pandharipande for the bielliptic locus is sharp.

In this chapter we prove the following two new results.

Theorem 4.2. The cohomology class $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological for all $g+m \geq 12$, $0 \leq n \leq 2g-2$ and $g \geq 2$.

Theorem 4.3. The cohomology class $[\mathcal{B}_{g,0,2m}]$ is nontautological when $g+m=12$ and $g \geq 2$.

Theorem 4.2 reduces the genus for which algebraic nontautological classes on $\overline{\mathcal{M}}_g$ are known to exist from 16138 to 12. As far as the author is aware, Theorem 4.3 provides the first example of a nontautological algebraic class on $\mathcal{M}_g$.

The Proof

We will start by proving the following weaker result.

Proposition 4.4. We have
$$[\overline{\mathcal{B}}_{g,0,2m}] \notin RH^\bullet(\overline{\mathcal{M}}_{g,2m})$$
for $g+m=12$ and $g \geq 2$.

Proof. Let
$$i\colon \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \to \overline{\mathcal{M}}_{g,2m} \tag{$\dagger$}$$
be the gluing morphism that pairwise identifies the first $g-1$ points on the first curve with the first $g-1$ points on the second curve. In Lemma 4.5 we will prove that the restriction of $i^*[\overline{\mathcal{B}}_{g,0,2m}]$ to the interior $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ is a positive scalar multiple α of the class $[\Delta]$ of the diagonal. Let $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$ denote the normalization of $(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}) \backslash (\mathcal{M}_{1,11} \times \mathcal{M}_{1,11})$. It follows from the localization sequence
$$A^{10}(\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})) \longrightarrow A^{11}(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}) \longrightarrow A^{11}(\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}) \longrightarrow 0$$
that $i^*[\overline{\mathcal{B}}_{g,0,2m}] = \alpha \cdot [\Delta] + B$, with B supported on the image of $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$.

The class B admits a tautological Künneth decomposition by Lemma 4.6.i. Given a homogeneous basis $\{e_i\}_{i\in I}$ for $H^\bullet(\overline{\mathcal{M}}_{1,11})$ with dual basis $\{\hat{e}_i\}_{i\in I}$, the cohomology class of the diagonal can be written as
$$[\Delta] = \sum_{i\in I} (-1)^{\deg e_i} e_i \otimes \hat{e}_i.$$
In particular, since $H^{11}(\overline{\mathcal{M}}_{1,11}) \neq 0$, the diagonal $[\Delta]$ does not admit a tautological Künneth decomposition. Because the pullback of a tautological class along a (composition of) gluing morphisms admits a tautological Künneth decomposition by repeated application of Proposition 2.1.33, this shows that $[\overline{\mathcal{B}}_{g,0,2m}]$ is nontautological. $\square$

Lemma 4.5. Let $g+m=12$ and $g \geq 2$. The pullback of $[\overline{\mathcal{B}}_{g,0,2m}]$ to $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$, under the restriction j of the gluing map i defined in ($\dagger$), is a scalar multiple α of the class of the diagonal Δ.

Proof. Let η be the map $\mathcal{M}_{1,11} \to \mathcal{A}dm(g,1)_{2m}$ which maps a curve $(C, x_1, ..., x_{11})$ to the admissible cover which has as a source curve two copies of C glued together by rational bridges attached to the first $g-1$ points of each copy of C, as covering involution the bielliptic involution which switches around the two copies of C and has two fixed points on each of the rational bridges, and as target curve a single copy of C with a rational component attached to the

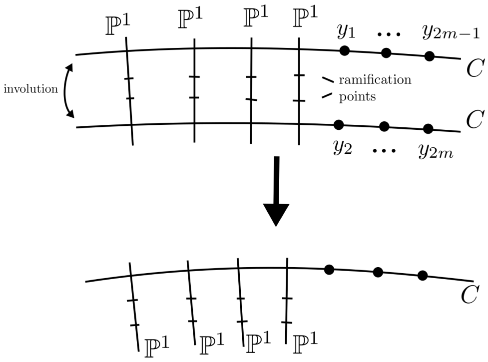


Figure 4.1: The image of C under η.

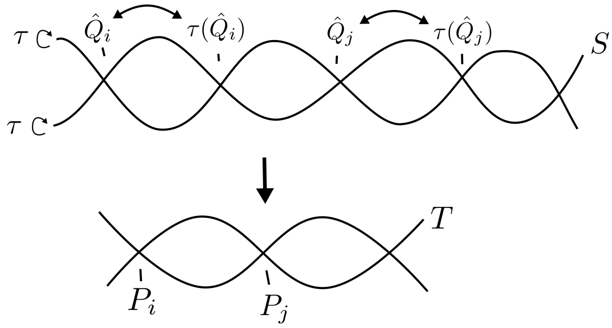


Figure 4.2: The case where τ fixes C_1 and C_2.

first $g-1$ points (see Figure 4.1). Let $\delta\colon \mathcal{M}_{1,11} \to \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ be the diagonal morphism. Consider the diagram

$$\begin{tikzcd}
\mathcal{M}_{1,11} \arrow[dr, dashed, "\zeta"] \arrow[drr, bend left, "\eta"] \arrow[d, bend right, "\delta"'] & & \\
\mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \arrow[r, "j"'] & F \arrow[r] \arrow[d, "\tilde{\phi}_0"] \arrow[dr, phantom, "\square"] & \mathcal{A}dm(g,1)_{2m} \arrow[d, "\phi_0"] \\
 & \mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \arrow[r, "j"'] & \overline{\mathcal{M}}_{g,2m}
\end{tikzcd} \qquad (\ddagger)$$

By unwrapping definitions one verifies that $j \circ \delta = \phi_0 \circ \eta$. By the universal property of fiber products this defines a unique map $\zeta\colon \mathcal{M}_{1,11} \to F$ making Diagram ‡ commute.

Claim: The morphism ζ is surjective on closed points.

Assuming the claim, it follows that $\tilde{\phi}_{0*}[F]$ is a positive scalar multiple of $\delta_*[\mathcal{M}_{1,11}] = [\Delta]$. Since

$$\operatorname{codim}_{\mathcal{M}_{1,11}\times\mathcal{M}_{1,11}} \Delta = 11 = \operatorname{codim}_{\overline{\mathcal{M}}_{g,2m}} \overline{\mathcal{B}}_{g,0,2m},$$

it follows that in Diagram ‡ there is no excess of intersection between $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ and $\overline{\mathcal{B}}_{g,0,2m} = \phi_0(\mathcal{A}dm(g,1)_{2m})$. We deduce that $j^*[\overline{\mathcal{B}}_{g,0,2m}] = \alpha[\Delta]$ for some $\alpha \in \mathbb{Q}_{>0}$.

Proof of the claim. By definition an object of $F(\mathbb{C})$ consists of a curve $\tilde{C} := (\tilde{C}_1, \tilde{C}_2) \in \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}(\mathbb{C})$, an object $(S \to T) \in \mathcal{A}dm(g,1)_{2m}(\mathbb{C})$ and an isomorphism $\gamma\colon j(\tilde{C}) \xrightarrow{\sim} \phi_0(S \to T)$. To prove the claim we will show that $(\tilde{C}, (S \to T), \gamma)$ is isomorphic to an object in the image of ζ. Let $f\colon \tilde{C}_1 \cup \tilde{C}_2 \to j(\tilde{C})$ be the map of curves induced by j, set $C := j(\tilde{C})$, $C_1 := f(\tilde{C}_1)$ and $C_2 := f(\tilde{C}_2)$, let τ be the involution on C induced by the bielliptic involution of $S \to T$ and let Q_i be the node of C corresponding to the i'th marking of $\tilde{C}_1$ and $\tilde{C}_2$ via the morphism f.

Since C_1 and C_2 are smooth, there are two possibilities for the action of τ on C: either it fixes C_1 and C_2 or it switches the whole of C_1 with the whole of C_2.

Suppose τ fixes C_1 and C_2. By construction the involution τ maps marked points lying on C_1 to marked points lying on C_2 so this is only possible if C has no marked points at all. In this case τ must fix the different branches of C at each Q_i. If the preimage of Q_i in S were to be a genus 0 curve R_i, contracted by the stabilization map, then R_i would have 2 marked ramification points which are not nodes, but this would imply that τ switches the nodes on R_i and it would

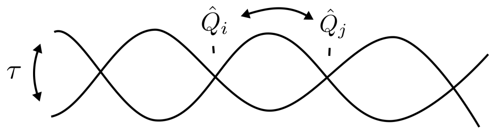


Figure 4.3: The case where τ permutes C_1 and C_2.

therefore also switch the branches of C at Q_i. It follows that the preimage of each Q_i in S is a single node $\hat{Q}_i$. Since C_1 and C_2 are smooth, τ induces an involution on the set of nodes $\{\hat{Q}_1, ..., \hat{Q}_{11}\}$. We can thus find distinct $\hat{Q}_i$, $\hat{Q}_j \neq \tau(\hat{Q}_i)$ such that $S - \{\hat{Q}_i, \tau(\hat{Q}_i), \hat{Q}_j, \tau(\hat{Q}_j)\}$ is connected. If P_i and P_j are the images of Q_i resp. Q_j under the admissible cover $S \to T$ then this means that $T - \{P_i, P_j\}$ is connected (see Figure 4.2). This implies that the arithmetic genus of T is at least 2, which is a contradiction.

We can therefore assume τ maps C_1 to C_2. Let us first suppose that τ does not fix all nodes, so there exist some distinct i, j such that $\tau(Q_i) = Q_j$ (see Figure 4.3). If the preimage of Q_i in S is a component of S contracted by the stabilization map then this component must contain a ramification point. This would be a fixed point of the involution, contradicting the assumption that $\tau(Q_i) = Q_j$. So the preimage of Q_i and Q_j in S are nodes $\hat{Q}_i$ and $\hat{Q}_j$. Let P be the image of $\{\hat{Q}_i, \hat{Q}_j\}$ under the bielliptic map. Arguing as at the end of the last paragraph we see that $T \backslash \{P\}$ is connected. Therefore, since T has arithmetic genus 1, it has geometric genus 0. However if S_1 is the irreducible component of S which surjects onto C_1 under the stabilization map then S_1 is a smooth curve of geometric genus 1. This is a contradiction because $S_1 \to T_1$ is a birational map.

We have thus proven that τ switches the components C_1 and C_2 and fixes the nodes Q_i, which implies that $((\tilde{C}_1, \tilde{C}_2), (S \to T), \gamma)$ is isomorphic to an object in the image of $\mathcal{M}_{1,11}(\mathbb{C})$. This concludes the proof that the map $\mathcal{M}_{1,11} \to F$ is surjective on closed points. □

Lemma 4.6. i Every algebraic class of codimension 11 in $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ supported on $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$ admits a tautological Künneth decomposition (see Definition 2.1.34).

ii Every algebraic class on $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ of codimension less than 11 admits a tautological Künneth decomposition.

Proof. This is a slightly weaker version of [GP03, Lemma 3], the proof given there requires that $RH^{2\bullet}(\overline{\mathcal{M}}_{1,n}) = H^{2\bullet}(\overline{\mathcal{M}}_{1,n})$ and $H^{\mathrm{odd}}(\overline{\mathcal{M}}_{1,n}) = 0$ for $n < 11$, for which there was no reference at the time of [GP03]. The first equation is [Pet14, Corollary 1.2]. The second condition follows from Getzler's computations for $n < 11$ in [Get98b]. □

We have now concluded the proof of Proposition 4.4. To prove Theorem 4.2 it remains to show that $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological for all g, n, m with $0 \leq n \leq 2g-2$ and $g+m > 12$.

Proof of Theorem 4.2. We will show in Lemma 4.7 and 4.8 that if $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological then so are $[\overline{\mathcal{B}}_{g,n+1,2m}]$ for $n \leq 2g-3$, and $[\overline{\mathcal{B}}_{g,n,2m+2}]$. In Lemma 4.9 we will show that if $[\overline{\mathcal{B}}_{g,1,0}]$ is nontautological then so is $[\overline{\mathcal{B}}_{g+1}]$. Using these statements, and by induction with base case the statement of Proposition 4.4, we conclude that $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological for all $g+m \geq 12$. □

Lemma 4.7. If $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological and $n \leq 2g-3$ then so is $[\overline{\mathcal{B}}_{g,n+1,2m}]$.

Proof. Let $\pi\colon \overline{\mathcal{M}}_{g,n+1+2m} \to \overline{\mathcal{M}}_{g,n+2m}$ be the morphism that forgets the first point and stabilizes. By definition $\pi_*([\overline{\mathcal{B}}_{g,n+1,2m}])$ is a positive scalar multiple of $[\overline{\mathcal{B}}_{g,n,2m}]$. Because the pushforward of a tautological class by the forgetful morphism is tautological by definition, the result follows. ☐

Lemma 4.8. If $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological then so is $[\overline{\mathcal{B}}_{g,n,2m+2}]$.

Proof. If $n \leq 2g-3$ then $[\overline{\mathcal{B}}_{g,n+1,2m}]$ is nontautological by Lemma 4.7. Consider the gluing morphism

$$\sigma\colon \overline{\mathcal{M}}_{g,n+2m+1} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n+2m+2}$$

which glues the first points of both curves together, then $\sigma^{-1}(\overline{\mathcal{B}}_{g,n,2m+2}) = \overline{\mathcal{B}}_{g,n+1,2m}$.

Since $\operatorname{codim}_{\overline{\mathcal{M}}_{g,n+2m+2}} \overline{\mathcal{B}}_{g,n,2m+2} = \operatorname{codim}_{\overline{\mathcal{M}}_{g,n+2m+1}} \overline{\mathcal{B}}_{g,n+1,2m}$ it follows that $\sigma^*[\overline{\mathcal{B}}_{g,n,2m+2}] = \alpha[\overline{\mathcal{B}}_{g,n+1,2m}]$ for some $\alpha \in \mathbb{Q}_{>0}$. Since σ is a gluing morphism and the pullback of a tautological class along σ admits tautological Künneth decomposition $[\overline{\mathcal{B}}_{g,n,2m+2}]$ is nontautological.

If $n = 2g-2$ we first prove that $[\overline{\mathcal{B}}_{g,n-1,2m+2}]$ is nontautological as above by pulling back along the map $\overline{\mathcal{M}}_{g,n+2m} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n+2m+1}$ and then applying Lemma 4.7. ☐

Lemma 4.9. If $[\overline{\mathcal{B}}_{g,1,0}]$ is nontautological then so is $[\overline{\mathcal{B}}_{g+1}]$.

Proof. Let $\epsilon\colon \overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{g+1}$ be the gluing morphism. From the description of the boundary divisors of $\overline{\mathcal{B}}^{\mathrm{Adm}}_{g+1}$ (see [Pag16, Page 1275-1276]) it follows that there exist $\alpha, \beta \in \mathbb{Q}_{>0}$ such that

$$\epsilon^*[\overline{\mathcal{B}}_{g+1}] = \alpha[\overline{\mathcal{B}}_{g,1,0} \times \overline{\mathcal{M}}_{1,1}] + \beta[(\overline{\mathcal{H}}_{g-1,0,2}, \overline{\mathcal{M}}_{1,1})] \in H^\bullet(\overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{1,1}),$$

where $(\overline{\mathcal{H}}_{g-1,0,2}, \overline{\mathcal{M}}_{1,1})$ denotes the locus of pairs $(C,E) \in \overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{1,1}$ where C consists of a genus $g-1$ hyperelliptic curve C' glued to an elliptic curve E' isomorphic to E, with the hyperelliptic involution switching the marked point of C' with the point of intersection with E'. The class $[(\overline{\mathcal{H}}_{g-1,0,2}, \overline{\mathcal{M}}_{1,1})]$ admits a tautological Künneth decomposition because the diagonal inside $\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1}$ does, the class of the hyperelliptic locus is tautological by [FP05, Theorem 1], and the pushforward of tautological classes under a gluing morphism is tautological by definition. The class $[\overline{\mathcal{B}}_{g,1} \times \overline{\mathcal{M}}_{1,1}]$ does not admit a tautological Künneth decomposition because $[\overline{\mathcal{B}}_{g,1}]$ is nontautological. It follows by Proposition 2.1.33 that $[\overline{\mathcal{B}}_{g+1}]$ is nontautological. ☐

We will now complete the proof of Theorem 4.3.

Proof of Theorem 4.3. The case $g = 2$ is treated in [GP03, Section 3]. We use a similar argument to prove the remaining cases. The proof runs by contradiction.

Suppose $[\mathcal{B}_{g,0,2m}] \in RH^\bullet(\mathcal{M}_{g,2m})$ then there is a collection of cycles Z_k in $\overline{\mathcal{M}}_{g,2m}$, of codimension 11 and supported on $\partial\overline{\mathcal{M}}_{g,2m}$, such that $\sum[Z_k] + [\overline{\mathcal{B}}_{g,0,2m}]$ is tautological. Consider again the gluing morphism $i\colon \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \to \overline{\mathcal{M}}_{g,2m}$ of (†). By assumption the pullback of $\sum[Z_k] + [\overline{\mathcal{B}}_{g,0,2m}]$ to $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ admits a tautological Künneth decomposition whereby the pullback of $\sum[Z_k]$ to $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ must be nontautological (by Proposition 2.1.33 and since the pullback of $[\overline{\mathcal{B}}_{g,0,2m}]$ is nontautological as we have shown in the proof of Theorem 4.2).

We shall denote by Δ_h the locus of curves in $\overline{\mathcal{M}}_{g,2m}$ consisting of two curves, one of which has genus h, glued together in a single node, and by Δ_{irr} the locus that generically parameterizes irreducible singular curves. So $\partial\overline{\mathcal{M}}_{g,2m} = \Delta_{\mathrm{irr}} \cup \bigcup_h \Delta_h$.

Suppose Z_k is supported on Δ_h for some h. Since $i(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$ does not have a separating node we see that $i(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}) \not\subset \Delta_h$. The intersection

$$\Delta_h \cap i(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$$

therefore lies in the image of $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$. It follows by Lemma 4.6.i that $i^*[Z_k]$ admits a tautological Künneth decomposition.

Suppose now that Z_k is supported on Δ_{irr}. We decompose the map i as

$$\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \xrightarrow{i_1} \overline{\mathcal{M}}_{g-1,2m+2} \xrightarrow{i_2} \overline{\mathcal{M}}_{g,2m}.$$

Then there exist cycles Y_k in $\overline{\mathcal{M}}_{g-1,2m+2}$ such that $i_{2*}[Y_k] = [Z_k]$. Now

$$i^*[Z_k] = i_1^* i_2^* [Z_k] = i_1^*(c_1(N_{\overline{\mathcal{M}}_{g-1,2m+2}} \overline{\mathcal{M}}_{g,2m}) \cap [Y_k]).$$

We see that $i^*[Z_k]$ decomposes as a product of algebraic classes of codimension less than 11, all of which admit tautological Künneth decomposition by Lemma 4.6.ii.

We conclude that all the cycles $[Z_k]$ have a tautological Künneth decomposition when pulled back to $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$, which is a contradiction. $\square$

Epilogue

Now that all the mathematics is done let me briefly discuss some of the questions that are still left open and highlight some areas where I think further study would be interesting.

This thesis has mainly been about spaces of admissible *double* covers. A natural question is whether we could have written our main results in the more general setting of cyclic covers or of Galois covers (or of even more general Hurwitz stacks of covers). The main reason for not working with more general covers was that the already considerable combinatorics of Chapter 2 is kept to a minimum. Working with double covers allows for a nice testing ground where many test calculations can be performed to check the general theory. However as far as I can see there is no fundamental obstruction to run what is basically the same argument in the more general case of Galois covers.

All the classes we compute in Chapter 3 are either classes of bielliptic loci or classes of hyperelliptic loci. However Theorems 2.2.21 and 2.2.26, our main theoretical underpinning for computing these classes, work for any space of admissible double covers $\mathscr{A}dm(g,h)_{2m}$. There is no obstruction to computing classes of the form $[\phi_n(\mathscr{A}dm(g,h)_{2m})]$ with $h > 1$ using the methods we described. I tried to compute a fair amount of classes to show how my methods work but the list of classes computed is by no means exhaustive.

Out of the methods presented in Chapter 3 to compute classes of spaces of admissible covers, the one of Section 3.4 is the only one that is not completely algorithmic. Indeed, in order to compute the intersection of the bielliptic locus with the hyperelliptic locus, we gave a set theoretic argument by combinatorial exhaustion over the boundary strata of $\overline{\mathcal{M}}_{0,2g+2}$. By further pushing this combinatorial exhaustion the pattern of Proposition 3.4.12 seems to hold (we will not state a proof here). More precisely it seems that the set theoretic inverse image of $\overline{\mathcal{B}}_g$ to $\overline{\mathcal{H}}_g$ consists of two components

- the locus of admissible hyperelliptic covers $S \to T$ where S is a curve with two irreducible components S_1 and S_2 of genus 2 and $g-3$ respectively and two nodes between them; and where S_1 admits a bielliptic involution switching the nodes,

- the locus of admissible hyperelliptic covers $S \to T$ where S is a curve with two irreducible components S_1 and S_2 of genus 1 and $g-2$ and two nodes between them; and where S_1 admits a bielliptic involution switching the nodes.

Granted this claim one could apply the method of Section 3.4 to the bielliptic locus in any genus. In light of the mistake made in [FP15] it would be good to have a more intrinsic proof of this statement.

One could even hope for a combinatorial formula in terms of spaces of admissible covers, gluing morphisms and ψ and κ classes for the intersection between any two spaces of admissible covers. We would then have formulas for intersections between two decorated stratum classes, between decorated stratum classes and spaces of admissible covers, and between two spaces of admissible covers.

The only algebraic cohomology classes which I know to be nontautological involve spaces of admissible covers. It would then be interesting to see if one could write down an extension $\hat{R}^\bullet(\overline{\mathcal{M}}_{g,n})$ of the tautological ring which includes these classes but is still tractable in the sense of having an additive set of generators and combinatorial formulas for the intersection between two such generators.

In the case where a range of classes of admissible covers can be shown to be tautological it is interesting to see if one can write down a recursive formula for such classes in terms of decorated stratum classes. Such a description is given in [CT17] for classes of the form $[\phi_n \mathcal{A}dm(2,0)]$ in $R^\bullet(\overline{\mathcal{M}}_{2,n})$. In this thesis we have not attempted to write down such formulas but it is conceivable that this result could be generalized to higher genus. Another range of classes of this form which one could try to write down a recursive formula for is $[\mathcal{A}dm(1,1)_{2m}] \in RH^{2\bullet}(\overline{\mathcal{M}}_{1,2m}) = H^{2\bullet}(\overline{\mathcal{M}}_{1,2m})$.

The results of Chapter 4 might hold for a much wider range of classes of admissible covers. We proved that $[\overline{\mathcal{B}}_{12}]$ is nontautological by pulling it back along the gluing map $\overline{\mathcal{M}}_{1,11}\times\overline{\mathcal{M}}_{1,11} \to \overline{\mathcal{M}}_{12}$. The pullback of $[\overline{\mathcal{B}}_{12}]$ consists of the class of the diagonal $\Delta \subset \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ together with a class D supported on the boundary of $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$. From analyzing the cohomology of $\overline{\mathcal{M}}_{1,11}$ we deduced that the class D admits a tautological Künneth decomposition. From the existence of odd cohomology on $\overline{\mathcal{M}}_{1,11}$ it follows that the diagonal does not admit a tautological Künneth decomposition. Because the pullback along a gluing morphism of a tautological class admits a tautological Künneth decomposition this means that the class of $\overline{\mathcal{B}}_{12}$ is nontautological.

One could try to run this argument in a more general setting. Any time we have a space $\overline{\mathcal{H}}$ of admissible (not necessarily double) covers such that one of the components of the pullback of $[\overline{\mathcal{H}}]$ along a gluing morphism is a diagonal $\Delta \subset \overline{\mathcal{M}}_{g,n}^{\times k}$ with g, n sufficiently high for there to exist odd cohomology on $\overline{\mathcal{M}}_{g,n}$, one would expect $[\overline{\mathcal{H}}]$ to be nontautological. However the proof of Proposition 4.4 breaks down in most other cases. There are several reasons for this: firstly the diagonal might not be of the expected dimension, secondly the pullback of $[\overline{\mathcal{H}}]$ usually contains a number of classes D_i different from the diagonal. In general if $\overline{\mathcal{M}}_{g,n} \neq \overline{\mathcal{M}}_{1,11}$ we have no easy way of showing that these classes D_i have a tautological Künneth decomposition. Therefore potentially the sum of these classes D_i and the class $[\Delta]$ of the diagonal might turn out to admit a tautological Künneth decomposition (although it would require a considerable amount of magic for this to happen).

Another interesting question is whether the range of classes proven to be nontautological in Proposition 4.4 is minimal in either of the following ways: is 22 the minimal k such that there exist nontautological algebraic classes in $H^k(\overline{\mathcal{M}}_{g,m})$? and is there a nontautological algebraic class in $H^{2k}(\overline{\mathcal{M}}_{g,m})$ for which $2g + m < 24$? The statement of Proposition 4.4 is the optimal result that can be achieved using the odd cohomology of $\overline{\mathcal{M}}_{1,11}$ and pullback along a sequence of gluing morphisms (in the sense of having the lowest codimension k and lowest value of $2g+m$ for which an algebraic nontautological class is known to exist in $H^k(\overline{\mathcal{M}}_{g,n})$).

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